MECHANICS OF LIQUIDS

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Archimedes (287–212 B.C.)—Eminent Greek scientist and mathematician, who first stated the law of buoyancy, and worked many complicated problems in stability of flotation.

Simon Stevin (1548–1620)—Flemish engineer and mathematician. Discovered the fundamental law of fluid pressure, and explained the hydrostatic paradox.

Evangelista Torricelli (1608–1647)—Italian scientist. Discovered that the velocity of a jet is $\sqrt{2\,g\,h}$.

Daniel Bernoulli (1700–1782)—Swiss scientist. Announced the theorem which bears his name in 1738

MECHANICS OF LIQUIDS

An Elementary Text in Hydraulics and Fluid Mechanics

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PREFACE

This book is a revision of a preliminary edition prepared in the summer of 1938, and used in our classes at Ohio State during the past year. Considering the large number of texts already available in this field, a word as to the reasons for writing this one may be in order.

During the past few years, there has been a great deal of discussion of *Fluid Mechanics* versus *Hydraulics*. Thinking of these as two competing subjects is a great mistake, in the writer's opinion. The question is very closely analogous to the corresponding situation in the mechanics of materials (solids). The teacher of Strength of Materials does not consider the Theory of Elasticity as a competing subject, but as the primary control with which he must constantly check. The reasons why he does not attempt to teach all this underlying theory are the practical ones of the limitations of time and the mathematical preparation of the majority of his students. Fluid Mechanics is not yet as completely worked out as Elasticity, but it has already developed far beyond the capacity of the usual American undergraduate.

The writer believes that with the time available at most institutions, and with the possible exception of a few specialized curricula, it is wisest to limit the first course to non-compressible liquids—to what is here called the Mechanics of Liquids. This differs very little from the old Hydraulics except in point of view and in the introduction of the improvements that have been developed by men like Osborne Reynolds, L. Prandtl, Th. von Kármán, and J. Nikuradse.

The fact that all liquids are viscous must be the cornerstone of the treatment, but experience has led the writer to believe that the best pedagogical order is to work from the simple to the complex, and to begin by treating those many cases where variations in viscosity have little effect. In fact, he has found it possible to confine the quantitative study of viscosity to the last three of the eight chapters. This leads to practically the historical order, starting with the pioneering work of Archimedes, Stevin, Torri-

celli, and the Bernoullis, and finishing with the work of the moderns mentioned above. An effort has been made to emphasize this historical approach.

The usual first chapter of discussion of the field and of the physical properties of liquids has been omitted with the idea that the student can best learn these as he progresses. Realizing the short time available in most institutions, a great effort has been made to be brief, with references in the footnotes to more complete treatment elsewhere. There has been no attempt to make this a text in Hydraulic Engineering, and the emphasis has always been placed on the mastery of principles, rather than on detailed facts and rules of practice necessary in actual engineering work. In the examples and problems, numerical values have often been chosen to facilitate computation rather than to simulate practice, so that the student may give his major attention to the method of solution rather than to precise values of constants. On the other hand, the writer has been guided in his selection of problems by his own experience as an hydraulic engineer.

Almost every article includes one or more worked out examples, and many facts are introduced into these and the problems without previous mention in the text, a practice he realizes all may not approve. In this as in many other things, he has attempted to follow the example of Professor James E. Boyd, under whom he had the pleasure of working for some years.

The writer wishes to thank Dean H. P. Hammond of The Pennsylvania State College, who read the entire manuscript and made many helpful suggestions. He wishes also to acknowledge aid given by his colleagues P. W. Ott, S. B. Folk, J. H. Koffolt, S. R. Beitler, E. C. Clark, LeRoy Tucker, and D. C. Williams. He wishes also to thank Professor J. K. Finch of Columbia University for the portrait of Chezy, and Professor Jasper O. Draffin of the University of Illinois for most of the other portraits, and Mr. C. V. Youngquist of the U.S.G.S. for help with the material on stream gaging and for photographs. The drawings are by Professor H. H. Brittingham, and speak for themselves.

RALPH W. POWELL

COLUMBUS, OHIO,
October 4, 1939.

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NOTATION

(Greek letters are grouped together following the English.)

A = area.
A_j = area of jet.
A_o = area of orifice.
A_1 = area at section 1, etc.
a = acceleration; half the height of an orifice.
B = width; length of weir crest.
b = width of rectangle or triangle.
$\mathbf{C}$ = Cauchy number $= V\sqrt{\frac{\rho}{K}}$.
C = a coefficient; Chezy coefficient; circumference.
C_c = coefficient of contraction.
C_d = coefficient of discharge.
C_v = coefficient of velocity.
C_1, C_2, and C_3 = coefficients used in economic pipe diameter problems.
c = celerity of pressure wave in pipes.
D = diameter; vertical dimension of rectangular orifice; depth of flow; mean depth of displacement.
D_a = average of D_1 and D_2.
D_c = critical depth.
D_n = normal depth.
d = sign of differential; altitude of rectangle or triangle.
E = Young's modulus; energy.
E_v = bulk modulus of elasticity.
e = base of Napierian logarithms; efficiency.
$\mathbf{F}$ = Froude number $= \frac{V}{\sqrt{gD}}$
F = total force.
F_x = x component of force, etc.
f = coefficient of "pipe friction."
$f(\ )$ = a function of ( ).
G = weight of a dam.

g = acceleration of gravity.
H = head, usually total; product of inertia; a horizontal force.
H_B = distance of center of buoyancy below center of gravity.
H_g = height of metacenter above center of gravity.
H_1 = head at point 1, etc.
h = head, usually partial; differential head in Venturi; subscript indicating horizontal.
$\bar{h}$ = head on center of gravity.
h_A = head at A, etc.
h_f = loss of head due to "friction"; total head lost.
h_j = head lost in jump.
h_o = assumed loss of head in pipe networks problems.
h_p = pressure head.
h_{p1} = pressure head at point 1, etc.
h_v = velocity head $\dfrac{V}{2g}$ or $\dfrac{\alpha V^2}{2g}$.
h_{va} = mean velocity head of approach above weir crest level.
h_{vb} = mean velocity head of approach below weir crest level.
I = moment of inertia of an area; impulse.
I_g = moment of inertia with respect to gravity axis.
J = ratio of D_2 to D_1 in hydraulic jump.
K = a constant, coefficient, or ratio; apparent bulk modulus.
K' = coefficient in orifice formula.
k = a constant; a coefficient; radius of gyration; number of fundamental units.
k_g = radius of gyration of area with respect to gravity axis.
k', k_1, k_2 and k_3 = coefficients in pipe formulas.
L = length.
l = length.
M = moment; momentum; momentum per second.
m = mass; mass per second; an exponent in pipe formulas; subscript indicating model.
N = normal force.
n = Kutter or Manning coefficient; an exponent in pipe formulas; number of quantities.
P = power, usually horsepower.
P_w = wetted perimeter.
p = unit pressure; subscript indicating prototype.
p' = unit pressure at a second point.

Q = volume of flow or discharge per second; a force.
Q_o and Q'_o = assumed flow in pipe networks.
q = discharge per foot of width.
$\boldsymbol{R}$ = Reynolds number $= \dfrac{VD}{\nu}$.
R = hydraulic radius; resultant force; radius of rounding.
r = ratio, such as throat diameter to pipe diameter; distance of point from center of pipe; radius of curvature; subscript indicating scale ratio of that quantity.
r_o = radius of pipe.
S = slope.
S_c = critical slope.
s = specific gravity; distance.
T = torque.
t = time.
t_1 = time to fall one foot.
t_w = thickness of pipe wall.
U = total uplift per foot of dam.
u = absolute velocity.
u_* = "friction velocity."
V = velocity; mean velocity; a vertical force.
V_A = velocity at A, etc.
V_t = tangential velocity.
V_6 = velocity in 6-inch pipe, etc.
v = velocity at a point, subscript indicating vertical; volume.
v_m = maximum velocity in cross-section.
v_o = displaced volume.
v_w = velocity at edge of boundary layer.
$\boldsymbol{W}$ = Weber number $= V\sqrt{\dfrac{\rho L}{\sigma}}$.
W = total weight; weight per second.
w = weight per unit volume.
x = horizontal or longitudinal distance from a point or axis; an unknown.
y = distance from an axis, or from wall of pipe.
y' = distance to center of pressure.
$\bar{y}$ = distance to center of gravity.
z = elevation above a datum.
z_A = elevation at A, etc.

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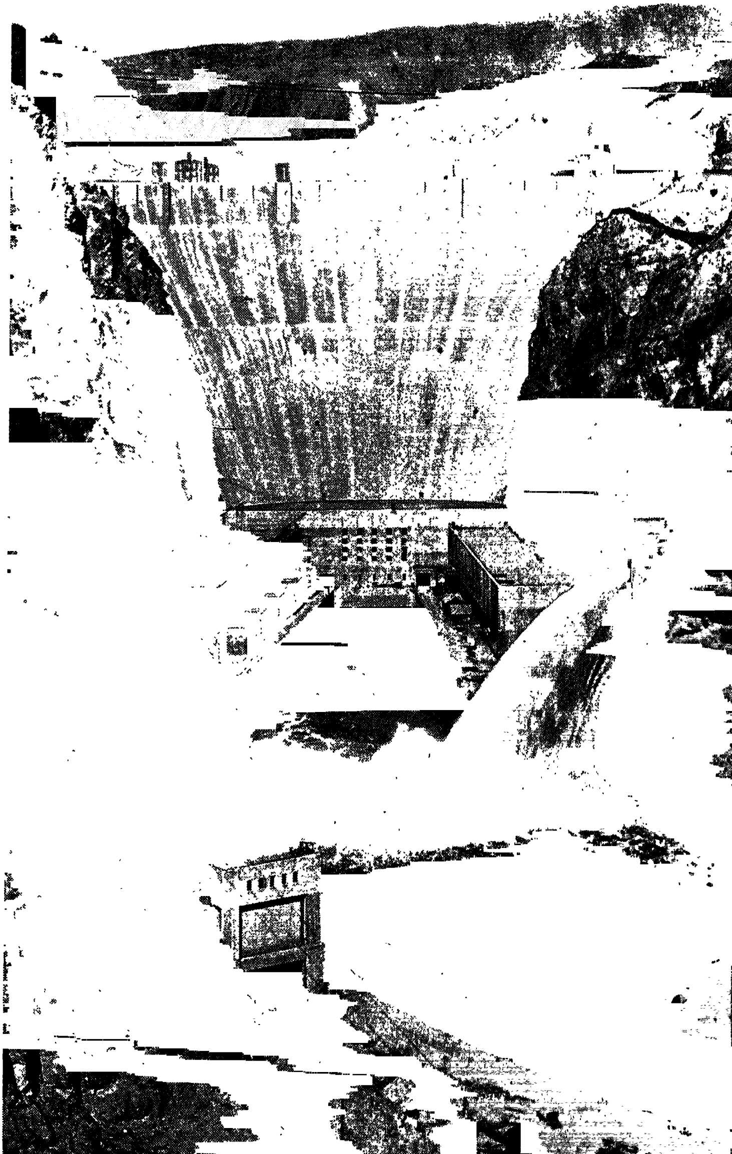


Fig. 1.—Boulder dam and power plant.
(Courtesy of Allis-Chalmers Mfg. Co.)

MECHANICS OF LIQUIDS

CHAPTER I

HYDROSTATICS

1. Buoyancy.—The problem of the pressure exerted by a liquid upon a body immersed in it, and upon the walls of the containing vessel, was one which puzzled scientists for centuries. The first steps in its solution were made by one of the greatest minds of antiquity, Archimedes.[1] His most important discoveries in this field are contained in the following propositions from Book I of his work *On Floating Bodies.*

"(2) The surface of any fluid at rest is the surface of a sphere whose center is the same as that of the earth. (3) Of solids those which, size for size, are of equal weight with a fluid will, if let down into the fluid, be immersed so that they do not project above the surface, but do not sink lower. . . . (5) Any solid lighter than a fluid will, if placed in the fluid, be so far immersed that the weight of the solid will be equal to the weight of the fluid displaced. (6) If a solid lighter than a fluid be forcibly immersed in it, the solid will be driven upwards by a force equal to the difference between its weight and the weight of the fluid displaced. (7) A solid heavier than a fluid will, if placed in it, descend to the bottom of the fluid, and the solid will, when weighed in the fluid, be lighter than its true weight by the weight of the fluid displaced."

Archimedes followed each of his propositions with a proof, in rigorous geometrical form, based on the following postulate:

"Let it be assumed that a fluid is of such a nature that, of the parts of it which lie evenly and are continuous, that which is pressed the less is driven along by that which is pressed the more, and each of its parts is pressed by the fluid which is perpendicularly above it except when the fluid is shut up in anything and pressed by something else."

[1] Born 287 B.C. and died 212 B.C. at Syracuse in Sicily. The translations are by Sir Thomas L. Heath, as printed in *The Works of Archimedes*, Cambridge University Press (1897), and *A History of Greek Mathematics*, Clarendon Press (1921), Vol. II p. 92.

What this means is illustrated by his proof of Proposition (3). Fig. 2 represents a part of the cross-section of the earth, with center at O, and fluid surface (say the Pacific Ocean), $ABCD$. $EFGH$ is a solid of the same specific gravity as the fluid, but assumed to float partly above the surface. He assumes a pyramidal portion of the fluid OLM, and an equal pyramid OMN, with $STUV$ equal and similar to the part $BGHC$ of the solid. Then he thinks of these pyramids as cut off by the spherical surface PQR. "Then the pressures on PQ, QR are unequal, that on PQ being the greater. Hence the part at QR will be set in motion by that at PQ, and the fluid will not be at rest; which is contrary to the hypothesis. Therefore, the solid will not stand out above the surface. Nor will it sink further, because all the parts of the fluid will be under the same pressure."

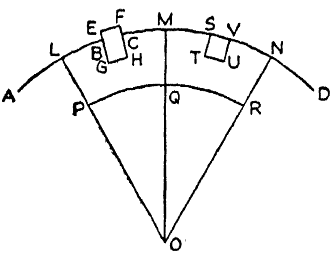


FIG. 2.

This last point is open to question, because if the solid is moved (by some outside force) to a position below the surface, it will still be in equilibrium. But in general, the reasoning power displayed by Archimedes in his treatment of buoyancy, marks him as one of the world's great thinkers, and centuries ahead of his time.

PROBLEMS

1–1. A hollow steel cylinder is 4 ft. outside diameter and 10 ft. long. It is partly filled with water so that the total weight of shell and contained water is 7000 lb. If it floats with its axis vertical in water weighing 62.5 lb. per cu. ft.,[2] how much will project out of the water? *Ans.* 1.09 ft.

1–2. What force would be required to immerse the cylinder of the preceding problem completely? *Ans.* 854 lb.

[2] More accurate values of the weight of water are given in Appendix A, Art. 66. If the water is pure and the temperature known, these accurate values should be used, at least to three significant figures. If the water is pure but the temperature is unknown or variable, 62.4 lb. per cu. ft. is probably the best value to use. In most engineering work 62.5 is close enough for practical purposes and because of its convenience, has been used in most of the problems of this text. Sea water weighs about 64.0 lb. per cu. ft., and the water in some salt lakes still more.

1-3. On the basis of Archimedes' postulate and by a method similar to that used in proving Proposition (3) above, prove Propositions (5), (6), and (7).

2. Pressure within a Liquid.—Practically no progress was made in unraveling the puzzles of hydrostatics for nearly 1800 years after the death of Archimedes. The next step forward was made by Simon Stevin,[3] a Dutch engineer (1548–1620). He realized that if a portion of a liquid at rest were thought of as solidified without change of density, the solid would be in equilibrium. Therefore, the upward pressure on the lower surface of the solid must exceed the downward pressure on the upper surface by the weight of the solid, that is, by the weight of the displaced liquid. But the surrounding liquid exerts just the same pressure on the solid whatever its density; therefore, any immersed body is buoyed up by a force equal to the weight of the displaced liquid. The same is true of a floating body. (To put it in everyday language, the liquid does not know what it is pressing against, and presses just the same whether it is a solid object or simply some more of the liquid.) Thus he proved Archimedes' Propositions (3) to (7) given in Art. 1.

Stevin was also the first to explain the "hydrostatic paradox" that if a long small vertical tube is fitted into the top of a strong

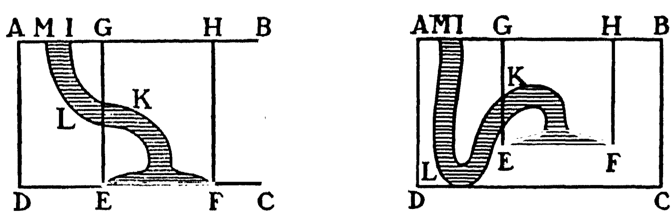


Fig. 3.—Drawings used by Stevin. (Courtesy Columbia University Press.)

barrel filled with water, the barrel can be burst by pouring an insignificant weight of water into the tube. His explanation was that the pressure at any point depends only on the depth below the free surface, and not at all on the area at the free surface. This he showed by first letting the sides of the vessel be vertical.

[3] The translation of the more important parts of his work on hydrostatics is given in Appendix I of the *Physical Treatises of Pascal* by Spiers and Barry, Columbia University Press (1937). His name is sometimes given in the Latinized form, Stevinus.

Then the total pressure on the bottom of the vessel is obviously just the weight of the liquid. But now suppose most of the upper part of the liquid to be solidified, leaving only a narrow neck extending up to the free surface. The pressure on the base will not be changed, because equilibrium is not disturbed by solidifying a portion of the liquid. In fact this does more than explain the paradox. It derives the law of fluid pressure given below (2.2). Figure 3 gives samples of the sort of drawings Stevin used in his reasoning.

To be more general, suppose the liquid of uniform specific weight w (for example, w pounds per cubic foot) at rest with a free surface AB, as shown in Fig. 4. (The fluid could not be a

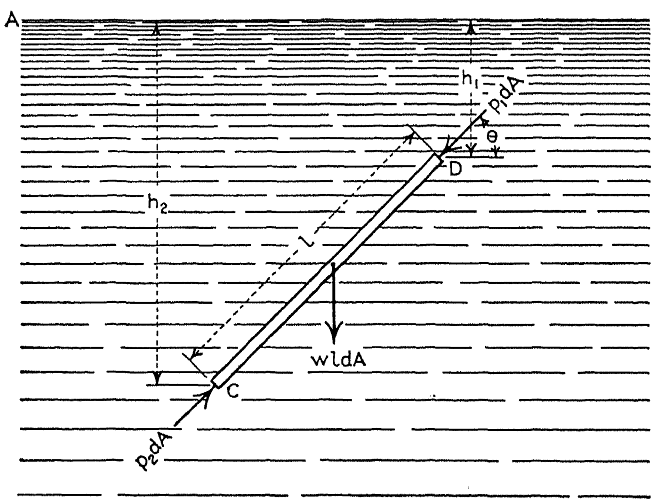


FIG. 4.—Thin prism of liquid considered as free body.

gas, because a gas has no free surface.) A certain portion of this liquid, CD, in the shape of a thin prism of cross-sectional area dA and length l, inclined at an angle θ with the horizontal, is taken as a free body. Since it is at rest, the forces on the free body must balance. Its weight will be w times its volume, and the component of the weight in the direction of the length of the prism will be $w\,l\,dA \sin\theta$. Let the unit pressure at C (that is, the pressure per unit area) be p_2 and the unit pressure at D be p_1. Then the total pressures on the ends will be $p_2\,dA$ and $p_1\,dA$ respectively. Besides these forces there will be the normal

pressure over the lateral surface. But if we write a resolution equation in the direction of the length, this is eliminated and

$$p_2\, dA = p_1\, dA + w\, l\, dA \sin\theta$$

and since $l \sin\theta = h_2 - h_1$,

$$p_2 - p_1 = w\,(h_2 - h_1) \tag{2.1}$$

It should perhaps have been specifically stated that there is no shearing stress along the walls of the prism. A liquid is a substance which will continue to deform as long as any shearing stress exists in it; therefore, in a liquid at rest there can be no shear.

By letting h_1 and p_1 be zero, and omitting the subscripts 2, we get

$$p = w\,h \tag{2.2}$$

which might be called Stevin's Law. It can be applied not only to a body of liquid with a free surface as in Fig. 4, but also to the case of a tank filled with a liquid under pressure as in Fig. 5. By dividing the pressure at any point by w, we get the height of liquid that would have caused that pressure, which is also the height to which the liquid would rise above that point in an open tube connected to it. This locates the imaginary free surface, and the pressure at any point in the liquid is given by (2.2) where h is the distance below this imaginary free surface. Attention should be called to the matter of units in (2.1) and (2.2). If w is in pounds per cubic foot and h in feet, p will be in pounds per square foot.

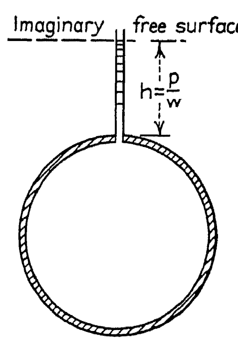


FIG. 5.

The unit pressures discussed above are all referred to the atmosphere as zero; that is, they are all what the engineer calls gage pressures. Absolute pressures are greater than gage pressure by one atmosphere (2116 pounds per square foot = 14.7 lb. per sq. in.). Therefore, the imaginary free surface for absolute pres-

sure will, for water, be higher than that for gage pressure by $\frac{2116}{62.5} = 33.9$ feet. (34 feet is often used for this value.) In most engineering practice gage pressures are to be understood unless it is stated to the contrary. If absolute pressures are used, the abbreviation "abs." is often placed after the numerical value of the pressure.

Formulas (2.1) and (2.2) can also be used for gases by use of an imaginary free surface and an assumed constant specific weight, as illustrated in Example II below. Actually, of course, the atmosphere gets lighter as we ascend above sea level and does not have a definite "air surface" like the "water surface" of the ocean.

EXAMPLE I

Figure 6 illustrates a closed cylinder 1 ft. long and 9 in. inside diameter, to which is attached a tube 1 in. in diameter. The whole is filled with water to a height of 41 ft. above the base. What is the total weight of water? What is the total pressure on the lower end of the cylinder?

Assuming the water to weigh 62.5 lb. per cu. ft., the total weight is $62.5\left(\frac{\pi 9^2}{4 \times 144} + \frac{40\pi}{4 \times 144}\right) = \frac{121\pi \times 62.5}{576} = 41.2$ lb. But the unit pressure on the bottom by (2.2) is 62.5×41 2562 lb. per sq. ft. = 17.80 lb. per sq. in. Therefore the total pressure on the bottom is $17.80 \times 81\pi$ 1132 lb. The difference between these two figures constitutes the hydrostatic paradox. The explanation comes when we consider the upward pressure on the top of the larger cylinder. It has an area of $\frac{81\pi}{4} - \frac{\pi}{4} = 20\pi$ sq. in. and is under a unit pressure of $\frac{62.5 \times 40}{144}$ 17.36 lb. per sq. in. Therefore, the upward pressure is $17.36 \times 20\pi = 1091$ lb. The difference is 41 lb., the weight of the water. (The slight discrepancy is due to carrying out some of the computations further than others.)

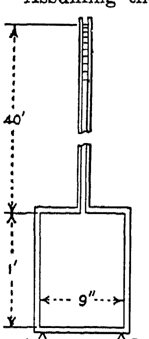


FIG. 6.

EXAMPLE II

A horizontal cylindrical tank 4 ft. in diameter is filled with air under a pressure of ten atmospheres absolute. What is the total outward pressure on one end? What is the net outward pressure? Where is the imaginary free surface if air at atmospheric pressure weighs 0.0765 lb. per cu. ft.?

Ten atmospheres = 21,160 lb. per sq. ft., therefore the total outward pressure = $21{,}160 \times 4\,\pi$ = 266,000 lb. The inward pressure will be one-tenth of this or 26,600 lb. and the net outward pressure will be the difference or 239,000 lb.

Air under a pressure of 10 atmospheres will weigh 0.765 lb. per cu. ft. and $h = \dfrac{p}{w} = \dfrac{21{,}160}{0.765}$ = 27,700 ft. = 5.26 miles. This result is, of course, independent of the pressure. At one atmosphere w is 0.0765 and h is still 5.26 miles. This is called the height of the "homogeneous atmosphere." Actually, of course, the atmosphere extends much higher than this, but at a decreasing density.

PROBLEMS

2–1. A rectangular tank, 3 ft. square on the bottom and 8 ft. high, has $1 \times 3 \times 4$ ft. portions in the upper corners filled in, as shown in Fig. 7. The tank is then filled with water weighing 62.5 lb. per cu. ft. What is the total weight of water in the tank, and what is the total water pressure on the bottom? Explain the difference. *Ans.* 3000 lb.; 4500 lb.

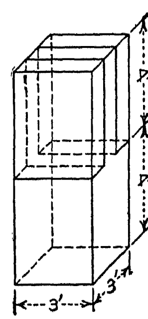


Fig. 7.

2–2. In Example I show that the difference between the upward and downward pressures is really *exactly* equal to the weight of the water. (Write the expressions in their original form and divide out the common terms.)

2–3. In Example I, if the metal shell enclosing the water weighs 100 lb. (including the pipe), what is the total pressure on the supports A and B? *Ans.* 141.2 lb.

2–4. If 10 lb. of the 100 lb. is the weight of the bottom plate, what is the total vertical force in the side walls of the 9 in. cylinder just above the lower end? Tension or compression? *Ans.* 1001 lb.

2–5. Figure 8 represents the so-called "Pascal's Vases" as shown in his *Traité de l'équilibre des liqueurs et de la pesanteur de la masse de l'air*, written in 1653. The vessels are all the same size at the base and are filled with water to the same depth. Will the pressure on the five stoppers be the same or different? Explain.

2–6. In Example II, if the pressure at the highest part of the cylinder is exactly 10 atmospheres, what will be the pressure at the lowest point? *Ans.* 10.0014 atmospheres.

2–7. When w is variable (2.2) must be written $\dfrac{dp}{dh} = w$. In the case of a perfect gas at uniform temperature where $w = \dfrac{p\,w_0}{p_0}$, show that $h_2 - h_1 = \left(\dfrac{p_0}{w_0}\right) \log_e \left(\dfrac{p_2}{p_1}\right)$.

2–8. Using the result of Prob. 2–7, find how far above the surface of the

earth the pressure would be one-tenth what it is at the surface *if* we could assume constant temperature. *Ans.* 12.06 miles.

2-9. Consider the elementary portion of a liquid shown in Fig. 9 to be so small that its weight can be neglected in comparison with the forces acting

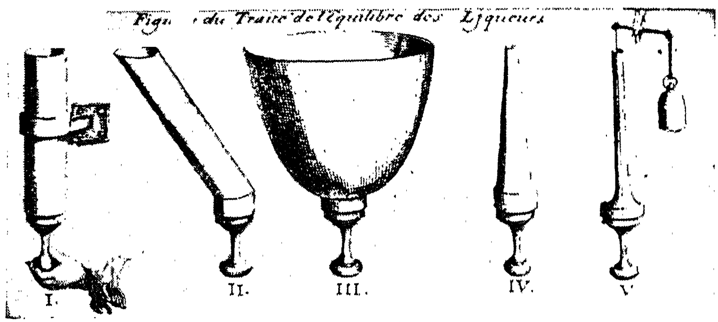


FIG. 8.—Pascal's vases. (Courtesy Columbia University Press.)

on it. Let the unit pressure on the top be p and that on the left side be p'. Write the expression for the total pressure on these two surfaces, and then by resolving parallel to ds (which eliminates N), show that $p = p'$. Then by resolving parallel to N find the unit pressure on the inclined face. Generalize your result.

3. Manometers.—In engineering practice pressure is measured in a number of different ways. Probably the simplest is actually to attach a tube as in Fig. 5 and observe directly the height to which the liquid will rise. Such a tube is called a *piezometer*. The bore at the liquid surface should be at least one-half inch to avoid serious capillary effects. Professor A. H. Gibson [4] says that the capillary rise of clean water in a clean glass tube D inches in diameter, is $\frac{0.0462}{D}$ inches, and that the depression of mercury is $\frac{0.0158}{D}$ inches. However, small amounts of impurities greatly change these quantities, so that it is safest to keep the correction small by avoiding too small a tube, and then if possible to arrange to measure differences, so that the cor-

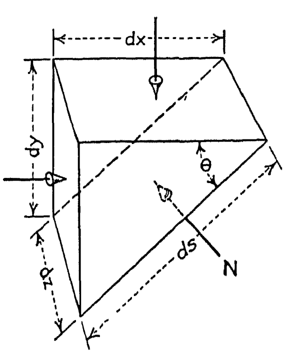


FIG. 9.

[4] *Hydraulics and Its Applications*, 4th Ed., Constable & Co., London (1930).

rections will cancel out. Pressure, especially when measured by a piezometer, is often called *head.* That is, we speak of a pressure or head of so many feet of water, or so many inches of mercury. Such units may be used even when measuring the pressure of some other fluid, such as illuminating gas.

Since such piezometer tubes might be inconveniently long (or in some cases, on the contrary, the head might be too small to measure accurately) a more common instrument for measuring pressures is the *manometer.* In its simplest form, this is a bent tube open to the air and containing some liquid heavier than the fluid whose pressure is being measured, and a vertical scale by which the difference of elevation in the two arms can be measured. In Fig. 10 suppose that the liquid in the tank and in the tube as far as A is water, and that ABC is mercury with a specific gravity of 13.6.[5] Then if C is 1 foot above A, and B is at the same level as A, it is obvious that the pressure at B is greater than the pressure at C by 13.6×62.5 pounds per square foot, or $\frac{13.6 \times 62.5}{144} = 5.90$ pounds per square inch, and the pressure at A being the same as at B is 5.90 pounds per square inch (gage). The pressure at any point in the tank at the same level as A will also be the same. Expressed in feet of water it will be 13.6 feet. If the bottom of the tank is 1 foot lower than A, the pressure on the bottom of the tank will be 14.6 feet of water.

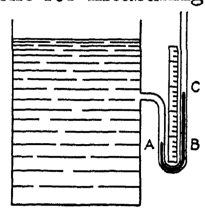


FIG. 10.—Open tube manometer.

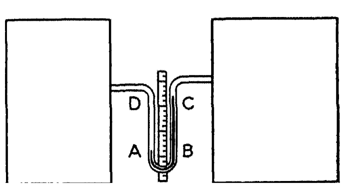


FIG. 11.—Differential manometer.

Very often the difference of pressure at two points is desired. In this case a *differential manometer* such as shown in Fig. 11 is used. If again the liquids in the two tanks are water and ABC is mercury of specific gravity 13.6, and if C is 2 feet above A, the pressure at A will exceed the pressure at C by 2 feet of mercury = 27.2 feet of water. But the pressure at D is 2 feet of water

[5] This is the value at 32° F. At ordinary room temperatures, mercury is only 13.57 times as heavy as the same volume of water at the same temperature. At higher temperatures the ratio increases again, being 13.60 at 100° F., and 13.93 at 212° F.

less than at A so that the pressure at D exceeds the pressure at C by only 25.2 feet of water. Therefore, the pressure at any point in the left hand tank exceeds the pressure in the right hand tank at the same level, by 25.2 feet of water, which is 12.6 times the gage reading. In other words, the column BC is 13.6 times as heavy as the column AD and the unbalanced pressure is $13.6 - 1 = 12.6$ times the pressure due to AD. Note that the position of the gage and its scale has no effect when differences of pressure are measured, whereas its position is important when a single pressure is measured, as indicated by the preceding paragraph.

When the difference of pressure to be measured is large, mercury is generally used in the manometer. On the other hand, when the difference is small, a liquid of nearly the same weight as that being measured is used. If it is heavier, the form of Fig. 11 is still used; but if it is lighter, the form is changed to that of Fig. 12. Why? If in Fig. 12 the liquid in the tanks is water and that in ABC is oil of specific gravity 0.80, and if A is 2.50 feet higher than C, the pressure at C will be $0.80 \times 2.50 = 2.00$ feet of water greater than at A. But the pressure at D is 2.50 feet of water greater than at A, therefore, the pressure at D exceeds the pressure at C by $2.50 - 2.00 = 0.50$ foot of water, and the pressure at any point in the left tank exceeds that at the same level in the right tank by 0.50 foot of water. Since the observed reading was 2.50 feet this arrangement has a multiplying factor of 5. By using a liquid of nearly the same weight as water, a very large multiplying factor may be obtained.

Fig. 12.—Differential manometer.

Fig. 13.—Bourdon pressure gage, assembled without dial to show internal mechanism. (Courtesy Crosby Steam Gage & Valve Co.)

In manometers such as these, if the bore is uniform and the liquid in the two tanks is the same, no correction for capillarity

is needed. Even here, however, small tubes should be avoided because of their sensitiveness to the presence of small impurities. It is also very important to keep the tubes free from air. Pressure differences so great that even a mercury column would be inconveniently long, are generally measured by Bourdon pressure gages. See Fig. 13. The essential element in these is a flattened tube, bent into a spiral. When the pressure in the fluid inside the tube increases it tends to "unwind" the spiral which moves the indicating needle of the instrument. Such a gage must be calibrated experimentally.

EXAMPLE

In Fig. 11 let both tanks contain oil weighing 50 lb. per cu. ft. and let ABC be water weighing 62.5 lb. per cu. ft. If C is 2.00 ft. above A, what is the difference in elevation in the free surfaces in the two tanks, and which is higher?

The pressure at A exceeds the pressure at C by 2.00 ft. of water $= 62.5 \times 2.00 = 125$ lb. per sq. ft. Since $\frac{125}{50} = 2.50$, this equals 2.50 ft. of oil. But the pressure at A exceeds the pressure at D by 2.00 ft. of oil; therefore, the pressure at D exceeds the pressure at C by 0.50 ft. of oil, and the free surface in the left tank is 0.50 ft. higher than that in the right tank. The multiplying factor of this arrangement is 4.

PROBLEMS

3–1. In Fig. 10, with water in the tank and mercury in the manometer, and with the zero of the scale 0.60 ft. above the bottom of the tank, the left mercury surface is at 0.50 ft. and the right one at 2.00 ft. on the scale. How high is the free water surface above the bottom of the tank, and what is the pressure on the bottom of the tank in lb. per sq. in.?

Ans. 21.50 ft.; 9.33 lb./sq. in.

3–2. In Fig. 11, ABC is carbon bisulfide with a sp. gr. of 1.29. If A reads 0.22 ft. and C reads 2.80 ft., what is the difference in elevation of the free water surface in the two tanks, and which is higher?

Ans. 0.75 ft. higher in left tank.

3–3. In Fig. 12, ABC is olive oil with sp. gr. of 0.91, A reads 2.54 ft. and C reads 0.54 ft. What is the difference in elevation of the free water surfaces in the two tanks and which is higher? *Ans.* 0.18 ft.

3–4. In Fig. 14, the tanks are filled with water weighing 62.5 lb. per cu. ft., and the tube contains oil weighing 50 lb. per cu. ft. What is the pressure difference between A and B in ft. of water, and which is greater?

Ans. 6.4 ft.

3–5. Show that in a manometer like Fig. 11 with the same liquid in the two tanks, the difference in pressure at the same level in the two tanks, measured in ft. of that liquid, will be $h\ (s - 1)$, where h is the difference of elevation of the liquid surface in the two arms of the manometer, and s is the ratio of the density of the liquid in the manometer to that of the liquid in the tanks.

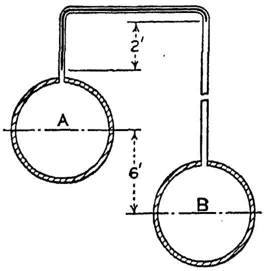


Fig. 14.

3–6. For Fig. 12, but with the other conditions as in Prob. 3–5, show that the corresponding difference is $h\ (1 - s)$.

3–7. In Fig. 11 the bottoms of the tanks and the zero of the scale are all on the same level. The left tank contains sea water weighing 64 lb. per cu. ft., the right tank contains fresh water weighing 62.5 lb. per cu. ft., and the manometer contains mercury weighing 846 lb. per cu. ft. A is at 1.00 ft. on the scale and C is at 1.50 ft. If the elevation of the free surface of the sea water is 19.0 ft. above the bottom of the tank, what is the elevation of the free surface of the fresh water? *Ans.* 13.16 ft.

4. Total Pressure on a Plane Area.—In Fig. 15, let $ABCD$ be any plane area in a liquid. The plane makes an angle of θ with the horizontal and intersects the surface of the liquid in

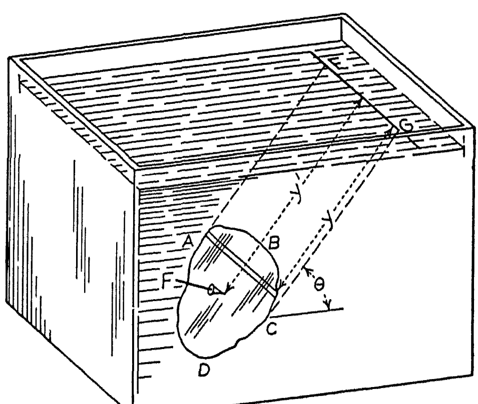


Fig. 15.—Inclined plane area in a liquid.

line EG. Now imagine the area divided into horizontal strips of infinitesimal width and area dA, and let y be the distance of a typical strip from line EG. Then the head on dA is $y \sin \theta$ and the unit pressure is $w\,y \sin \theta$. Then the total pressure on dA is $w\,y \sin \theta\, dA$, and the total pressure on the area will be

$$F = w \sin \theta \int y \, dA = w \sin \theta \, \bar{y} \, A$$

where $\bar{y}$ is the distance from the center of gravity of the area to line EG. But $\bar{y} \sin \theta = \bar{h}$ = the head on the center of gravity. Therefore the total pressure

$$F = w \, \bar{h} \, A \tag{4.1}$$

In other words, the average pressure on a submerged plane surface is equal to the pressure on its center of gravity, and the total, or resultant, pressure on the area is equal to this average intensity multiplied by the area. (This could be proved still more directly by measuring the distance of each strip vertically from the free surface instead of along the inclined plane, but the method here used avoids confusion with the next article. It fails when $\theta = 0$, but from (2.2) it is seen that (4.1) is correct for that case also.)

EXAMPLE

The triangular tank shown in Fig. 16 is 10 ft. wide and 15 ft. long, and 12 ft. deep. Find the total pressure on an end and on a side when it is full of water weighing 62.5 lb. per cu. ft.

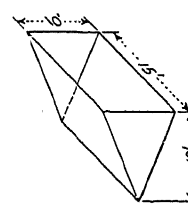


FIG. 16.

The center of gravity of an end is 4 ft. below the free surface and $A = \frac{10 \times 12}{2} = 60$ sq. ft. Therefore $F = 62.5 \times 4 \times 60 = 15{,}000$ lb.

The center of gravity of a side is half way down the incline which is 6 ft. below the free surface. $A = 15 \times 13 = 195$ sq. ft. $F = 62.5 \times 6 \times 195 = 73{,}100$ lb.

PROBLEMS

4–1. A rectangular tank, 5 ft. long, 4 ft. wide and 6 ft. deep, is filled with water. Find the total pressure on the bottom, on one side, and on one end. *Ans.* 7500 lb.; 5620 lb.; 4500 lb.

4–2. A triangular tank similar to Fig. 16 is 8 ft. long, 6 ft. wide at the top, and 4 ft. deep. Find the total pressure on one end and on one side when it is filled with oil weighing 50 lb. per cu. ft. *Ans.* 800 lb.; 4000 lb.

4–3. A cylindrical tank, with axis horizontal, is 4 ft. in diameter. It is connected to a 2-in. pipe, and the tank and pipe are filled with water to a height of 12 ft. above the axis of the tank. Find the total pressure on one end. *Ans.* 9420 lb.

4-4. A cylindrical tank with axis horizontal and 12 inches in diameter, is filled with mercury weighing 846 lb. per cu. ft., up to the center of the cylinder. Find the pressure on one end. If you have forgotten the location of the center of gravity of a semicircle, look it up. *Ans.* 70.5 lb.

4-5. A tank 8 ft. square and 4 ft. deep is filled with water. What is the total hydrostatic pressure on any one side? If the tank is held together by four horizontal hoops one foot apart, with the lowest 6 in. from the bottom, what is the tension in each hoop? (Assume that each hoop carries the pressure on the foot of height nearest it.)

Ans. 4000 lb.; 125 lb. in top hoop; 375 lb. in 2nd; 625 lb. in 3rd; 875 lb. in lowest.

NOTE. This illustrates the fact that although the resultant force equals the area times the unit pressure at the center of gravity, it does not act at the center of gravity. As shown in the next article, the 4000 lb. would in this case act at 16 in. from the bottom.

4-6. By resolutions and moments, find the magnitude and location of the resultant of the tensions in the four hoops on any one side of the tank in the preceding problem. Draw a clear sketch showing the forces involved.

Ans. 16.5 in. above the floor.

5. Center of Pressure.—The center of pressure on an area is the point of application of the resultant total pressure on the area. For example, in Fig. 15 it is the point where the total force F which acts on the area $ABCD$, intersects that area. Let its distance from EG be y'. It can be found by taking moments about EG. The force acting on dA is $w\,y \sin\theta\, dA$ and it has a lever arm of y, therefore its moment is $w\,y^2 \sin\theta\, dA$. Summing all such moments we have

$$F\,y' = w \sin\theta \int y^2\, dA = w \sin\theta\, I$$

But $F = w \sin\theta\, \bar{y}\, A$, therefore

$$y' = \frac{I}{\bar{y}\,A} = \frac{k^2}{\bar{y}} \tag{5.1}$$

where I and k are with respect to the axis EG. But with respect to a horizontal axis through the center of gravity let the radius of gyration be k_g. Then $k^2 = k_g^2 + \bar{y}^2$ and

$$y' = \frac{k_g^2}{\bar{y}} + \bar{y} \tag{5.2}$$

That is, the center of pressure is always below the center of

gravity by a distance $\frac{k_g^2}{\bar{y}}$ measured along the plane. Obviously as $\bar{y}$ gets large the center of pressure approaches the center of gravity. Formulas (5.1) and (5.2) apply to all cases except when $\theta = 0$. For this case of a horizontal surface, the center of pressure is, of course, at the center of gravity.

The most important case is that of a rectangular surface with one pair of edges horizontal. Then $k_g^2 = \frac{d^2}{12}$ where d is the non-horizontal dimension of the rectangle. If the rectangle extends to the free surface, $\bar{y} = \frac{d}{2}$ and $\frac{k_g^2}{\bar{y}} = \frac{d}{6}$ so that the center of pressure is two-thirds the way from the free surface to the bottom of the rectangle. (What about the case where the rectangle does not extend to the free surface?)

Generally the lateral position of the center of pressure is not required. If there is a vertical axis of symmetry as is usually the case, the center of pressure must, of course, lie on it. Even for certain unsymmetrical areas, such as parallelograms and trapezoids, the center of all the horizontal strips will lie on a straight line, and the center of pressure is on this line. For the general case, see Prob. 5–12.

EXAMPLE I

Locate the center of pressure in the example of Art. 4.

For a triangle $\frac{b\,d^3}{36}$ and $\frac{b\,d}{2}$, k_g^2 $\frac{d^2}{18}$, and $\frac{k_g^2}{\bar{y}}$ $\frac{d^2}{18} \times \frac{3}{d}$ $\frac{d}{6}$. Therefore, the center of pressure on the ends is 2 ft. below the center of gravity, or 6 ft. below the free surface. For the sides, as pointed out above, the center of pressure will be two-thirds the way down the incline or 8 ft. below the free surface.

EXAMPLE II

In Fig. 17, if the gate AB is 5 ft. wide perpendicular to the paper, find the reactions at A and B.

The force on the left side of AB is $62.5 \times 8 \times 4 \times 5 = 10{,}000$ lb. and its point of application is below the center of gravity,

$$\frac{k_g^2}{\bar{y}} = \frac{4^2}{12 \times 8} = \frac{1}{6} \text{ ft.}$$

The force on the right side is $62.5 \times 1.5 \times 3 \times 5 = 1406$ lb. and its point of application is 1 ft. above B. Then taking moments about B,

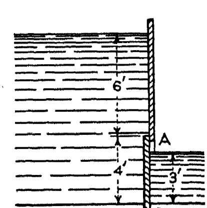


FIG. 17.

$$4R_A = 10{,}000 \times \frac{11}{6} - 1406 \times 1 \qquad \text{and} \qquad R_A = 4232 \text{ lb.}$$

Taking moments about A,

$$4R_B = 10{,}000 \times \frac{13}{6} - 1406 \times 3 \qquad \text{and} \qquad R_B = 4362 \text{ lb.}$$

$$4232 + 4362 + 1406 = 10{,}000 \text{ lb., which checks.}$$

PROBLEMS

5-1. Locate the centers of pressure in Prob. 4-2.
Ans. $y' = 2.00$ ft. on end and 3.33 ft. on side.

5-2. Locate the center of pressure in Prob. 4-3.
Ans. 1 in. below the center.

5-3. Locate the center of pressure in Prob. 4-4. (Start from the fact that the moment of inertia of a semicircle about its bounding diameter is $\frac{\pi r^4}{8}$.)
Ans. 3.53 in. below surface.

5-4. A vertical rectangular gate, 8 ft. wide and 9 ft. high, is subjected to water pressure on one side, the free surface being level with the top of the gate. The gate is hinged at the bottom and held by a horizontal chain at the top. Find the tension in the chain. *Ans.* 6750 lb.

5-5. Find the tension in Prob. 5-4 if there was water also on the other side of the gate to within 3 ft. of the top. *Ans.* 4750 lb.

5-6. Make a diagram showing the variation of pressure in Prob. 5-5 and show that for that part of the gate which has water on both sides, the unbalanced pressure is uniform. How much is it? Make a generalized statement of this case.

5-7. Find the center of pressure on a vertical rectangular gate, 12 ft. high, due to water which rises 3 ft. above the top of the gate. (This additional 3 ft. is held by some other support, as in Fig. 17. If the water were flowing over the top of the gate the pressure would be somewhat reduced and the center of pressure slightly lowered, as pointed out in Art. 21.)
Ans. 10.33 ft. below the water surface.

5-8. A navigation lock gate is 32 ft. wide and 40 ft. high. When the water on the upstream side is 27 ft. deep and that on the downstream side is 12 ft. deep, what is the resultant net hydrostatic pressure on the gate and where is it applied? Which item of data is not needed?
Ans. 585,000 lb.; 10.23 ft. above bottom.

5-9. Figure 18 represents the end of a cylindrical tank 4 ft. in diameter and full of water under pressure. The zero of the scale is level with the bottom of the tank and the mercury reads 1 ft. in the left tube and 3 ft. in the right tube. Taking the sp. gr. of the mercury as 13.6, find the total pressure on the end of the tank, and the location of the center of pressure.
Ans. 20,600 lb.; 0.0382 ft. below center.

5-10. In Fig. 19 the face of the dam slopes 4 ft. up to each 3 ft. horizontal. The gate is 12 ft. × 10 ft. (perpendicular to the paper). Its center is 24 ft. directly below the water surface. If the gate is supported entirely by its upper and lower edges, find the reactions.

Ans. 84,000 lb. and 96,000 lb.

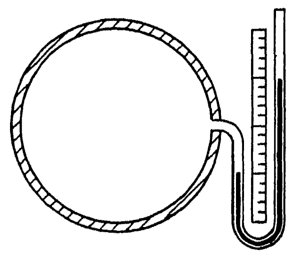


FIG. 18.

5-11. Solve Prob. 5-9 if the fluid in the tank, instead of being water, is a gas whose average weight is 0.150 lb. per cu. ft.

Ans. 21,400 lb.; about 0.001 in. below center.

5-12. By a method similar to that used above, but taking moments about an axis in the plane and perpendicular to EG (Fig. 15), show that $x' = \frac{H}{\bar{y}A}$, where H is $\int x\,y\,dA$, the product of inertia of the area with respect to EG and the axis just mentioned, and x' is the horizontal distance from the axis to the center of pressure.

5-13. A "butterfly" gate consists of a circular disc of approximately the same size as the pipe in which it is placed. Suppose such a gate pivoted on a vertical axis in a horizontal pipe 10 ft. in diameter. Find the location of the center of pressure on the right half of the gate when the head on the center of the gate is 25 ft.

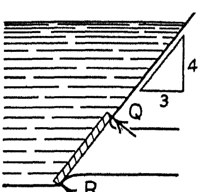


FIG. 19.

Ans. 2.12 ft. to the right of the vertical center line, and 0.25 ft. below the horizontal center line.

5-14. A triangular vertical area ABC has AB horizontal and 8 ft. long, and AC vertical with C 6 ft. below A. Locate the center of pressure if the free surface is at AB. Work by the formula, and check by the fact that the center of pressure must lie on the median running from C to the middle of AB. Why?

6. Pressure on Curved Surfaces.—Figure 20 represents a portion of a liquid enclosed by a curved surface ABC and an imaginary plane ADC at θ with the horizontal. The resultant pressure on the plane is F and the weight of the liquid is W. The resultant of the normal pressures on the curved surface may be broken into two components, parallel and perpendicular to F. Let the parallel component be Q. Resolving in that direction, the other component is eliminated, and

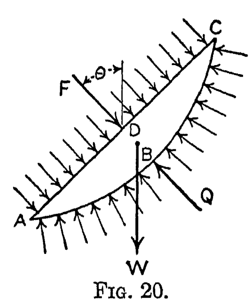


FIG. 20.

$$F + W\cos\theta = Q \tag{6.1}$$

If the plane is vertical

$$F = Q \tag{6.2}$$

In a gas this latter equation is also practically true for all values of θ, because W is too small to be considered. It is approximately true in a liquid under high pressure where F and Q will be much larger than W. Hence the rule that the component in any direction of the total pressure of a fluid on a curved surface, is equal to its total pressure on the projected area perpendicular to that direction.

EXAMPLE

Find the resultant pressure per ft. of length (perpendicular to the paper) on the lower left hand quarter of the cylindrical surface of the tank in Prob. 5–9. The drawing of a good sketch is left to the student.

Projecting on a vertical plane through the center of the tank, the horizontal pressure varies from 26.2 ft. of water at the center to 28.2 at the bottom. Therefore, the average pressure is $27.2 \times 62.5 = 1700$ lb. per sq. ft. The horizontal force per ft. of tank is $1700 \times 2 = 3400$ lb. As a rough approximation the vertical force would be the same and the resultant force would be $3400\sqrt{2} = 4810$ lb. The true solution is obtained by taking the lower left hand quarter of the water in 1 ft. of tank as the free body. The downward force on the top is $26.2 \times 62.5 \times 2 = 3275$ lb. and the weight of the water is $62.5\,\pi = 196$ lb. Therefore the downward component is 3471 lb., and the resultant force is $\sqrt{3400^2 + 3471^2} = 4859$ lb. Since the area is π sq. ft. this gives an apparent average normal pressure of 1547 lb. per sq. ft. which corresponds to 24.7 ft. of water. Actually the head varied from 26.2 to 28.2 ft. The discrepancy lies in the fact that the vector sum of the normal pressures is less than their scalar sum, since they are not parallel.

PROBLEMS

6–1. Draw a chord to the quadrant of the above example and find the pressure per linear ft. on the plane of which it is the trace.

Ans. 4810 lb.

6–2. In the above example find the line of action of the weight of the quadrant of water, and of the resultant downward force, and find the angle that the 4859 lb. force makes with the horizontal. Draw an accurate scale drawing showing where this force cuts the shell of the tank. Does it cut it at right angles? Investigate the general case.

One of the answers is 45° 35′.

6–3. Figure 21 represents a Tainter gate. If the radius of the cylindrical surface is 20 ft.; the length (perpendicular to the paper) 16 ft.; the pivot A, 10 ft. above the seat B; and the water surface just at the level of the pivot;

find the total horizontal thrust and the total vertical upward component of the water pressure. (Work first, neglecting the weight of water *BCD*, and then considering it.) What is the resultant water pressure? Does it go

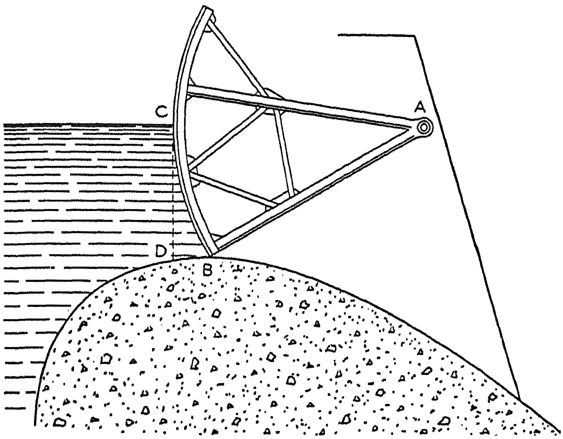


Fig. 21.—Tainter gate.

through *A*? (Assume the center of gravity of *BCD* to be directly above a point three-tenths of the way from *D* to *B*.)

Resultant water pressure = 53,200 lb.

7. Dams.—One of the important engineering applications of the subject of hydrostatics is in the design of dams. Books have been written on the subject, and it can only be touched on here. Figure 22 is a cross-section of the abutment section of the Grand Coulee dam. (This is the part over which water is never expected to flow. There is also a lower, spillway portion.) *A* is called the toe of the dam and *B* the heel. *F*, the total water pressure on the upstream face (generally taken per foot of length), can be figured in amount and location by (4.1) and (5.2) when the face is a plane. When it is curved, or broken as in Fig. 22, it is necessary to break this force up into its two components *H* and *V*. *V* is simply the weight of the wedge of water *BCDE*, and *H* equals the pressure on *BC* and acts at one-third the distance from *B* to *C*. The location of force *V* can generally be easily calculated from the given dimensions of the dam by dividing the area *BCDE* into triangles, or triangles and rectangles. Similarly, the amount and location of *G*, the weight of the dam, can be figured (given its dimensions and weight per cubic foot). Then point *J*, the inter-

section of the line of action of F and G can be found, either graphically or by computation, and then point I, the point where R, the resultant of F and G, intersects the base. Or this point may

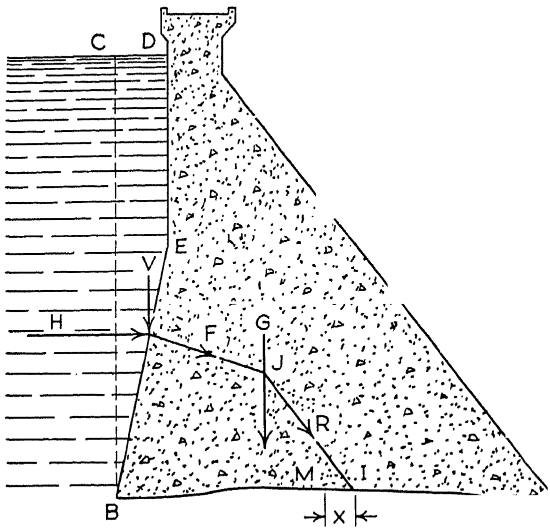


FIG. 22.—Grand Coulee dam.

be found directly by moments about any point in the base. For equilibrium the reaction at I must have a horizontal component equal to H and a vertical component equal to $V + G$. Then by taking moments about M, the midpoint of the

$$(7.1)\quad x\,(V + G) = H \times \text{its lever arm} - G \times \text{its lever arm} - V \times \text{its lever arm}$$

where x is the distance of point I to the right of M. As is shown in texts on Strength of Materials,[6] x must not be greater than one-sixth of the base of the dam, to avoid a reversal of stress at the heel. That is, point I must lie within the middle third of the dam. Also, if the width of the base of the dam is represented by B, we know from the Mechanics of Materials that the unit pressure on the foundation at the toe will be

[6] For example, Boyd's 4th Ed., McGraw-Hill Book Co. (1935), p. 367.

$$\frac{V+G}{B}+\frac{6\,M}{B^2} \tag{7.2}$$

and the unit pressure at the heel will be

$$\frac{V+G}{B}-\frac{6\,M}{B^2} \tag{7.3}$$

M here represents the total moment about the midpoint of the base, that is, either side of equation (7.1). This assumes a straight line variation in the foundation pressure, so that the pressure at any other point can be found by interpolation. The actual situation is probably never so simple, but this is the assumption usually made.

Another requirement is that the coefficient of friction times $(V + G)$ must be more than H, so that their ratio will give a proper factor of safety. If a preliminary design is found to violate this requirement, it is generally best to introduce more slope on the upstream face, as this increases both V and G. For this reason, in hollow dams where G is relatively less, the upstream slope is much greater than in solid dams.

The question of possible *uplift* must be considered in designing dams, and forms a good illustration of the facts of hydrostatics. If any water can seep from the reservoir under the dam, it will, if at rest, be under the full head of the reservoir. For example in Fig. 22 at a point 544 ft. below the water surface the hydrostatic pressure will be $62.5 \times 544 = 34{,}000$ lb. per sq. ft. or 236 lb. per sq. in. If this existed under the whole foundation it would counterbalance a large part of the weight of the dam, and it would no longer be safe against sliding. But, of course, it would not exist under the whole dam, as the cross-sectional area of the dam is somewhat more than half the base times the altitude, and concrete weighs more than twice as much as water. In other words, the dam will not float, and must rest directly on at least part of its foundation with no intervening film of water. It is usual to assume that uplift exists on from 0 to 50 or more per cent of the base area, depending on the nature of the foundation rock, precautions taken in construction, and the optimism or pessimism of the designing and consulting engineers. Another important point is that if water can seep under the heel it can

generally seep all the way to the toe, and will then find its way out into the tail water, and at exit its head will be only that due to the tail water. As will be shown in Chapter IV, if the cross-section area of the seepage channel is uniform throughout its

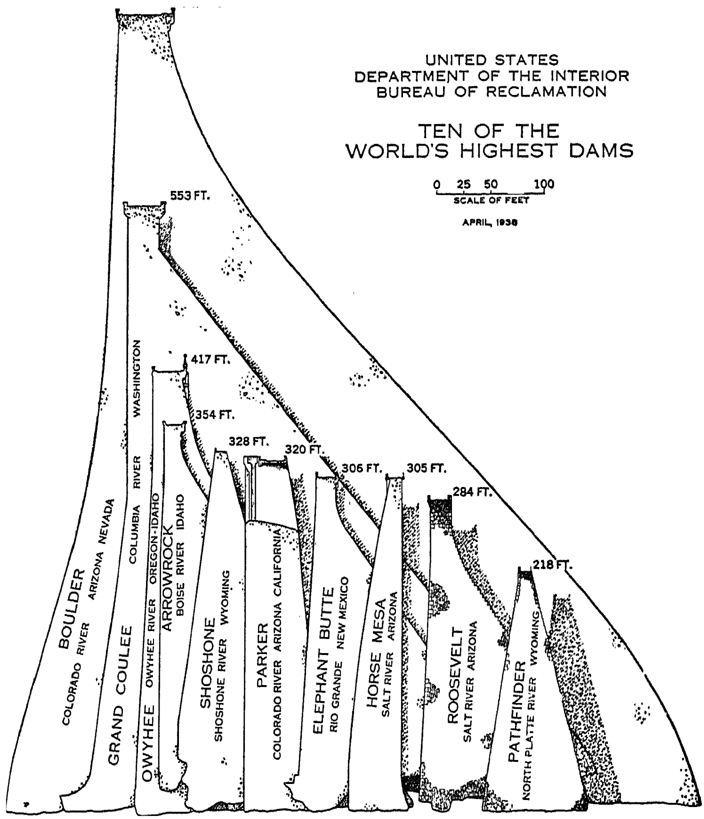


FIG. 23.—Sections of ten dams. Some are arch dams and would not be stable otherwise. (Courtesy of Bureau of Reclamation.)

length, the pressure will decrease at a uniform rate from heel to toe. This assumption is often made in design. It reduces the total uplift, but introduces an additional overturning effect, so that x in (7.1) will be increased. If U = total uplift force (per foot of length of dam) another positive term, $U \times B/6$, is introduced on the right side of (7.1).

EXAMPLE

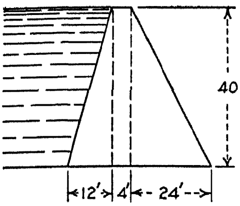


FIG. 24.

If the dam in Fig. 24 weighs 150 lb. per cu. ft., find the H and V components of the water pressure and the weight of the dam (all per ft. of length), and determine the factor of safety against sliding on the foundation if the coefficient of friction is 0.50. Also find the unit pressure on the foundation at toe and heel. Work first with no uplift, and then with uplift varying from full hydrostatic at the heel to zero at the toe, but acting over only 40 per cent of the base area.

$$H = 20 \times 62.5 \times 40 = 50{,}000 \text{ lb.}$$

$$V = \frac{12 \times 40}{2} \times 62.5 = 15{,}000 \text{ lb.}$$

$$G = 150\,(240 + 160 + 480) = 132{,}000 \text{ lb.}$$

$$V + G = 147{,}000 \text{ and } 0.5\,(V + G) = 73{,}500 \text{ lb.}$$

Therefore factor of safety against sliding $\dfrac{73{,}500}{50{,}000} = 1.47$. This may be considered sufficient. To find x it is best to split G up into its three parts, the weight of the heel triangle, the rectangle, and the toe triangle. They are 36,000, 24,000, and 72,000 lb. respectively. Then moments about the center of the base gives $M = 50{,}000 \times 13.33 + 72{,}000 \times 4 - 36{,}000 \times 12 - 24{,}000 \times 6 - 15{,}000 \times 16 = 138{,}500$ lb.-ft. Then the unit pressure at the toe is $\dfrac{147{,}000}{40} + \dfrac{6 \times 138{,}500}{1600}$ $3675 + 520 = 4195$ lb. per sq. ft., and the unit pressure at the heel is $3675 - 520 = 3155$ lb. per sq. ft. These are safe for rock.

Considering uplift, the pressure at the heel is $40 \times 62.5 = 2500$ lb. per sq. ft. and at the toe, zero. This is an average of 1250 lb. per sq. ft. 40 per cent of the area of the base would be 16 sq. ft., therefore $U = 20{,}000$ lb. This reduces the total vertical force to 127,000 lb. and the factor of safety to 1.27, which is hardly sufficient. The lever arm of the uplift about the midpoint would be $40/6 = 6.67$ ft., and the moment $= 20{,}000 \times 6.67 = 133{,}400$ lb.-ft. This increases M to 272,000 and $\dfrac{6\ M}{1600}$ becomes 1020. Then the unit pressure at the toe is $3175 + 1020 = 4195$ lb. per sq. ft., and at the heel it is $3175 - 1020 = 2155$ lb. per sq. ft. These are safe for rock.

PROBLEMS

7–1. A triangular dam is 48 ft. high and 33 ft. wide at the base and weighs 150 lb. per cu. ft. If the upstream side is vertical, with water clear to the top, find the necessary coefficient of friction to give a factor of safety of 1.5

against sliding (a) with no uplift, (b) with uplift varying from full hydrostatic pressure at the heel to zero at the toe, over 50 per cent of the base area.

Ans. 0.91; 1.15

7-2. Check the "middle third" requirement for Prob. 7-1 for (a) and for (b). *Ans.* Safe; unsafe.

7-3. Solve Prob. 7-1 if the dam were reversed with the vertical face downstream. *Ans.* 0.64; 0.75

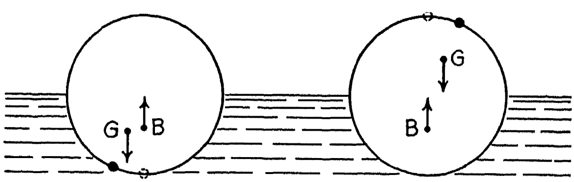
7-4. Do the same for Prob. 7-2. *Ans.* Unsafe.

7-5. Solve the example with uplift as given, but over 60 per cent of the base.

7-6. Solve the example with the uplift as given over 40 per cent of the base but with the 12 ft. dimension reduced to 6 ft.

Ans. Not enough factor of safety against sliding.

8. Stability of Flotation.—As Archimedes discovered, a floating solid will sink until it displaces its own weight of the liquid. The resultant of all the pressures which the liquid exerts on the surface of the solid is an upward force equal to the weight of the displaced liquid and to the weight of the solid. If the solid is in equilibrium, these two forces will act along the same line. If the solid is displaced slightly from its position of equilibrium,



(a) Stable (b) Unstable

Fig. 25.—Stable and unstable equilibrium.

however, the two forces will generally no longer act along the same line, but will form a couple. If the original position was one of *stable* equilibrium, this couple will tend to return the solid to its original position. If the displacement produces no couple, the equilibrium is said to have been *neutral*, while if it produces a couple which tends to make the solid rotate still further in the same direction, the equilibrium was *unstable*. Figure 25 illustrates the cylindrical drum with keel described in Prob. 8-4. G is the center of gravity of the drum, and B is the center of gravity of the displaced water. In (a) the keel was originally at the bottom. Rotation through a small clockwise angle produces a counter-

clockwise couple tending to bring it back to its original position. In (b) the keel was originally at the top. Rotation through a small clockwise angle produces a clockwise couple tending to turn it to position (a). If there had been no keel the center of gravity would have been at the center of the drum and always just above the center of the displaced water, and there would have been no couple.

The buoyant force always acts directly upward through the center of buoyancy, which is the center of gravity of the displaced liquid. The center of buoyancy may be above or below the center of gravity of the solid, or may coincide with it. It may seem at first sight that if the center of gravity is above the center of buoyancy, the equilibrium will be unstable. But this is not usually the case, as will be seen in the following examples and problems. When a floating body is tipped, there will be a wedge-shaped portion of it rising out of the liquid on one side, and a corresponding wedge immersed on the other. The latter has buoyant effect tending to raise the side that went down, and the former makes extra weight on the side that went up, so that there will ordinarily be a righting moment, even when the center of gravity is above the center of buoyancy.

EXAMPLE

A rectangular scow of reinforced concrete weighing 150 lb. per cu. ft. is 50 ft. long, 20 ft. wide, and 10 ft. high, with walls and bottom 1 ft. thick. Locate its center of gravity and its center of buoyancy when floating in water

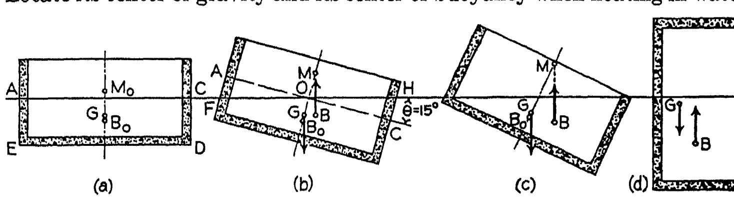


FIG. 26.—Stability of flotation of rectangular scow.

weighing 62.5 lb. per cu. ft. Also compute the righting moment for various angles of list, assuming that there is a watertight deck, whose weight may be neglected.

$$
\begin{aligned}
\text{Volume of side walls} &= 2\,(50 + 18)\,10 = 1360 \text{ cu. ft.}\\
\text{Volume of bottom} &= 48 \times 18 \qquad\quad = \underline{864 \text{ cu. ft.}}\\
& \qquad\qquad\qquad\qquad\;\; 2224 \text{ cu. ft.}
\end{aligned}
$$

$$\text{Height of center of gravity above bottom} = \frac{1360 \times 5 + 864 \times 0.5}{2224} = \frac{7232}{2224}$$

$= 3.25$ ft. The weight of the scow is $2224 \times 150 = 333{,}600$ lb., therefore the displacement will be $333{,}600 \div 62.5 = 5338$ cu. ft. Then it will sink $\dfrac{5338}{50 \times 20} = 5.34$ ft. into the water, and the center of buoyancy will be 2.67 ft. above the bottom. Figure 26 shows four positions, (a) that of stable equilibrium, (b) a list of 15°, (c) a list of 25°, and (d) a list of 90°.

The righting moment in case (b) is computed as follows: The distance

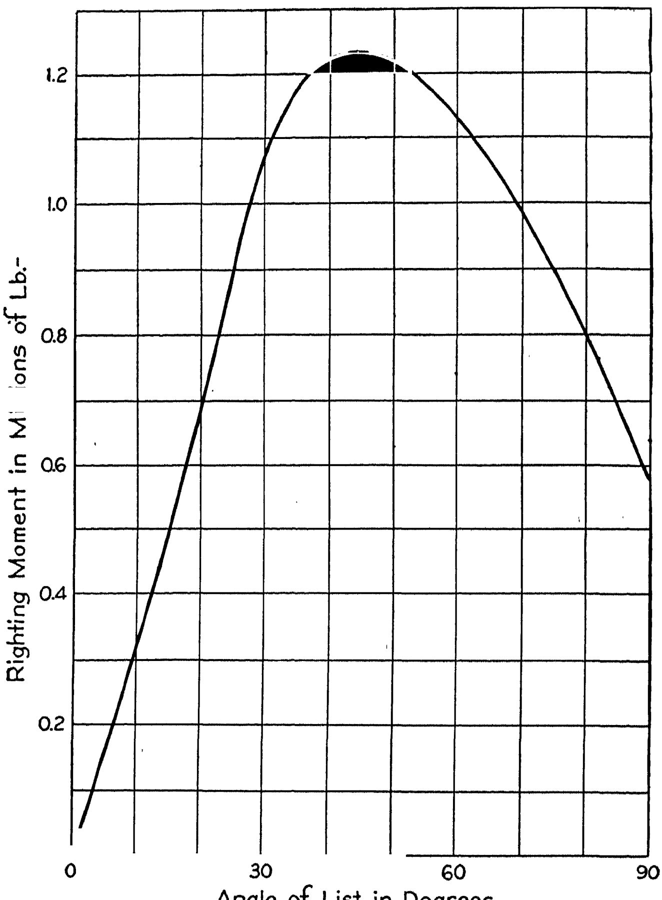


FIG. 27.—Righting moments for scow of Fig. 26.

that B_0, the original position of the center of buoyancy, was below G, the center of gravity, was $3.25 - 2.67 = 0.58$ ft. Then if the displaced portion of the liquid had remained *ACDE*, there would have been a clockwise moment of $333{,}600 \times 0.58 \times \sin 15° = 50{,}300$ lb.-ft. But the buoyancy is in-

creased by the immersion of the wedge OCH, and decreased because the wedge OAF lifts out of the water. This produces a counterclockwise couple of $62.5 \times 50 \times 10 \times 10 \tan 15^\circ \left(\frac{20}{3} \cos 15^\circ + \frac{10}{3} \tan 15^\circ \sin 15^\circ\right)$ $312{,}500 \times 0.2680\ (6.44 + 0.23) = 558{,}600$ lb.-ft. Then the net righting moment is $558{,}600 - 50{,}300 = 508{,}300$ lb.-ft. Similarly the righting moments for 25° and 90° are computed and plotted in Fig. 27.

Generalizing the method of the above example, with L as the length, B the width, W the total weight, H_B the distance B_0 is below G, θ the angle of list, and M the righting moment, we have

$$M = \frac{w\,L\,B^2 \tan\theta}{4}\left(\frac{B\cos\theta}{3} + \frac{B\tan\theta\sin\theta}{6}\right) - W\,H_B \sin\theta.$$

If D is the distance the solid sank in the liquid when in the stable position of equilibrium, and the total weight of the displaced liquid $W = w\,D\,L\,B$, then $M = \frac{W\,B^2 \sin\theta}{24\,D}(2 + \tan^2\theta) - WH_B \sin\theta$, and the horizontal distance between the force of gravity acting down from G, and the buoyant force acting up through point B, is $\frac{M}{W} = \frac{B^2 \sin\theta}{24\,D}(2 + \tan^2\theta) - H_B \sin\theta$. Line $\overline{B_0G}$ produced is called the *normal vertical.* It was vertical when the solid was in its stable position, and moves with the solid as it tips. In any tipped position, the intersection of the line of action of the bouyant force with the normal vertical is indicated on the figure as point M. Then the distance $\overline{GM}$ is $\frac{B^2}{24\,D}(2 + \tan^2\theta) - H_B$. Therefore, the distance $\overline{B_0M}$ is $\frac{B^2}{24\,D}(2 + \tan^2\theta)$. As θ approaches zero, this approaches

$$\overline{B_0M_0} = \frac{B^2}{12\,D} \quad \frac{k^2}{D} \quad \frac{I}{v_0} \tag{8.1}$$

This distance is called the *metacentric height,* and point M_0 is called the *metacenter.* It is the limiting position which the intersection of the buoyant force and the normal vertical approaches, as the angle of list approaches zero. In equation (8.1), k is the radius of gyration of the "water-line area" about its longitudinal horizontal axis. Further investigation would show that the equa-

tion holds, not only for a rectangular solid, but for any shape which has vertical sides at the water line. In the general case, D will be the mean depth of displacement; that is, the displaced volume, v_0, divided by the waterline area.

If the metacenter is below the center of gravity, the equilibrium is unstable; and if it is at the center of gravity, it is neutral. If the metacenter is above the center of gravity, equilibrium is stable, but sometimes when the metacenter is very high, point M falls rapidly as θ increases, so that the stability is limited to small lists, and the craft is treacherous. It must be realized that whenever a torque is suddenly applied to a floating vessel, as for example by the shifting of cargo, or by a change in the external liquid level on one side, as when a "swell" strikes a rowboat broad-side, the vessel will, due to its inertia, turn through a greater angle than that corresponding to the torque shown by a curve similar to Fig. 27. If inertia carries the vessel past the point of zero righting moment, it will capsize. But if it comes to rest while the righting moment is still positive, it will start back, and go past the equilibrium position. This produces the "roll" of a ship. The greater the metacentric height, the faster the roll. As a first approximation, the period of oscillation is inversely proportional to the square root of the height of the metacenter above the center of gravity.

A floating body can also, of course, rotate about a transverse horizontal axis. This is known as *pitching*. The metacenter for this motion will be much higher than for rolling. In spite of the fact that the radius of gyration with respect to the transverse axis will also be greater than that about a longitudinal axis, the time for a complete cycle of pitching in most ships is only about half that for rolling.[7]

PROBLEMS

8–1. Find the metacentric height in the above example. *Ans.* 6.24 ft.

8–2. Find the righting moment in the example for case (d), when $\theta = 90°$. *Ans.* 583,000 lb.-ft.

8–3. Find θ in the above example when the right-hand upper corner is just at the water line, and the righting moment for that position. *Ans.* 25°; 894,000 lb.-ft.

8–4. A hollow cylindrical drum 4 ft. outside diameter and 8 ft. long, weighs

[7] For further information, see article on "Shipbuilding" in *Encyclopaedia Britannica*, 14th Ed.

1571 pounds. A keel of the same weight as the drum is then attached along an element of the cylinder, partly inside and partly outside, so that its center of gravity is on the outer surface of the drum, and the drum floats in water with its axis horizontal. How far below the water surface will it be to (a) the center of the drum, (b) the center of gravity, and (c) the center of buoyancy? What is the metacentric height? *Ans.* 0; 1.00 ft.; 0.849 ft.; 0.849 ft.

8–5. In the preceding problem, what will be the righting moment when the drum rolls 45°? 90°? *Ans.* 2222 lb.-ft.; 3142 lb.-ft.

8–6. For the preceding problem, sketch a curve of righting moment against angle of roll from 0° to 180°, and note where the moment is a maximum.

8–7. A cylindrical caisson is 20 ft. in diameter and is closed at the bottom. It floats with its axis vertical and 20 ft. of its length below the water surface. What is the metacentric height? What is the maximum distance of the center of gravity from the bottom for stability? *Ans.* 1.25 ft.; 11.25 ft.

8–8. Continuing the curve of Fig. 27 beyond 90°, at what position is the righting moment zero? Is the body in equilibrium in that position? If so, of what sort? When θ is 180°, is it in equilibrium? Of what sort? What is the metacentric height for that position? *First ans.* 114° 43′.

8–9. In the example, write a general expression for the righting moment when θ is greater than, say, 30°, differentiate it with respect to θ to find where it is maximum, and find the maximum moment. Check against Fig. 27. *Ans.* 44° 18′; 1,236,000 lb.-ft.

8–10. In Dynamics we learn that the complete period of an angular vibration due to a torque which varies directly as the angular displacement, is $2\pi\sqrt{\frac{Wk^2}{gK}}$, where W is the weight of the rotating mass (in pounds), k is its radius of gyration about its axis of rotation (in feet), K is the ratio of torque (in lb.-ft.) to displacement in radians, and the time is in seconds. Assuming that the roll takes place about the center of gravity as a center, and that it is small enough so that the righting moment will be proportional to the list, show that the time of a complete rolling vibration will be $\frac{2\pi k}{\sqrt{g H_g}}$, where H_g is the height of the metacenter above the center of gravity.

8–11. Using the result of the preceding problem, find the approximate time of a small amplitude roll in Prob. 8–4. *Ans.* 1.92 sec.

8–12. Using the result of Prob. 8–10, find the approximate time of a small amplitude roll in the example. Take k as 3.15 ft. *Ans.* 1.47 sec.

8–13. Show that (8.1) is true for all vessels which in their position of equilibrium have their sides vertical at the water line.

8–14. If an object weighing 25,000 lb. were placed just touching the floor at the center of the scow of the example, and suddenly released, how far would the scow sink to a position of equilibrium? What would be its downward velocity when it reached this position, and how much further down would it go before stopping (neglecting friction)? What would be the period of this "dipping" oscillation?

Ans. 0.40 ft.; 0.947 ft./sec.; 0.40 ft.; 2.65 sec.

CHAPTER II

FUNDAMENTALS OF FLUID FLOW

9. Law of Continuity.—If a fluid enters a pipe from a tank through a well rounded entrance as in Fig. 28, it is found that near the entrance the velocity is nearly uniform across the cross-section. If all particles passing the section BB' were moving parallel to the axis of the pipe with a velocity V, and continued so to move for one second, at the end of one second they would all be at the section CC', a distance V ft. downstream. Obviously the fluid which passed the section BB' in this second is now all in the space $BB'CC'$ and therefore its volume is $A\ V$, where A is the cross-sectional area. In American practice the number of cubic feet of fluid which passes any cross-section per second is called the *discharge* or the *flow*, and is represented by the letter Q. Hence we have the fundamental formula

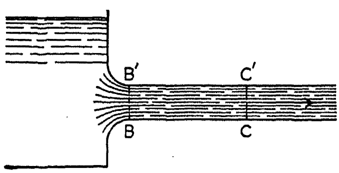


FIG. 28.—Pipe with rounded entrance.

$$Q = A\ V \tag{9.1}$$

Cubic feet per second is often abbreviated to c.f.s. Although considered incorrect by some, the expression "second foot" is also well established as an alternate name for this unit. The weight passing any section per second would be

$$W = w\,Q = w\,A\ V \tag{9.2}$$

and the mass [1] per second would be

[1] For those who have not used engineering units in dynamics, it should be pointed out that W/g will be in slugs (or gee pounds) per sec., where the slug is that mass which is accelerated 1 ft. per sec.[2] by a force of one lb. It is g lb. of mass. See any text on Engineering Dynamics, such as Boyd's *Mechanics*, 2nd Ed., McGraw-Hill Book Co. (1930), pp. 224–226.

$$\frac{W}{g} = \frac{w}{g} A\ V = \rho\ A\ V \tag{9.3}$$

where ρ (rho) is the density (mass per unit volume).

Downstream from the entrance the velocity will not continue to be uniformly distributed across the cross-section, because the walls of the pipe will retard the fluid next to them, and this layer will retard the next inside layer, and so on. But we can still speak of the average velocity, V, defining it as Q/A. And then, of course (9.1), (9.2), and (9.3) will still be true.

The *Law of Continuity* is simply the obvious statement that in steady flow without storage, what goes in at the upstream section must come out at the downstream section, so that

$$w_1 A_1 V_1 = w_2 A_2 V_2 \tag{9.4}$$

For a liquid, w_2 will be equal to w_1 and

$$A_1 V_1 = A_2 V_2, \text{ or } Q \text{ is constant.} \tag{9.5}$$

Since the pressure along a pipe generally varies, a gas flowing in a pipe will not have a constant w, and (9.5) will not apply to it.

The use of the law of continuity is in pipes or channels of varying cross-sectional area. By it, knowing the mean velocity at any one point where the area is known, we can compute the velocity at any other point where the area is known, or vice versa. Since most pipes are circular and the area is proportional to the square of the diameter, the mean velocity in pipes flowing full is given by

$$D_1^2 V_1 = D_2^2 V_2 \tag{9.6}$$

EXAMPLE

A 12-in. pipe reduces to a 6-in. diameter, and then expands to an 8-in. diameter. If the mean velocity in the 6-in. pipe is 16 ft. per sec., what is the mean velocity at the other sections?

Writing the velocity in the 12-in. pipe as V_{12}, we have $V_{12} = \dfrac{6^2 \times 16}{12^2} = \dfrac{16}{2^2}$ $= 4.00$ ft. per sec. Similarly, $V_8 = \dfrac{6^2 \times 16}{8^2} = 16\left(\dfrac{3}{4}\right)^2 = 9.00$ ft. per sec.

PROBLEMS

9–1. Solve the above example if the velocity in the 6-in. pipe is unknown, but the velocity in the 12-in. pipe is 6 ft. per sec.

Ans. 24.0 ft./sec.; 13.50 ft./sec.

9–2. Water flows full in an 8 ft. diameter pipe, and then empties into a trapezoidal channel, with 10 ft. bottom width and 1 to 1 side slopes, where it flows 4 ft. deep with a mean velocity of 5 ft. per sec. What was the mean velocity in the pipe? *Ans.* 5.57 ft./sec.

9–3. If the pipe of Fig. 28 continues downstream for some distance at the same diameter, will the velocity at the center of the pipe be more than, less than, or equal to, the mean velocity at section BB'?

10. Torricelli's Theorem.—A younger contemporary of Stevin, Galileo Galilei (Italian, 1564–1642) one of the great founders of modern science, gave a great deal of attention to the mechanics of liquids, but without much result. In 1630 he remarked that the laws controlling the motion of the planets in their celestial orbits were better understood than those governing the motion of water on the surface of the earth. And this is, in a sense, still true in 1939. But the year after Galileo's death one of his disciples, Evangelista Torricelli (Italian, 1608–1647) made an extremely important discovery. This was, that if water escaped from a sharp-edged opening in the side of a tank, as in Fig. 29, the jet had a velocity equal to that gained by a body falling the distance from the free surface to the orifice, that is,

(10.1) $$V =$$

This is known as Torricelli's theorem. We now realize that it

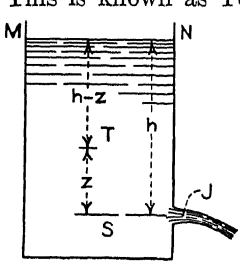


Fig. 29.

follows directly from the law of the conservation of energy and the definition of kinetic energy. And it applies to all liquids as well as water. If in Fig. 29 a liquid flows out of the orifice at a rate of W pounds per second, and we add liquid to the tank at exactly the same rate to maintain the head h, there will be a flow of liquid from the free surface MN to the jet J. Each W pounds of water so flowing, loses $W h$ foot-pounds of potential energy, but gains a kinetic energy in the jet of $m V^2$ $\frac{W V^2}{2 g}$ If no en-

ergy is lost these are equal. Hence

$$W\,h = \frac{W\,V^2}{2\,g} \qquad \text{and} \qquad V = \sqrt{2\,g\,h} \tag{10.1}$$

Careful measurements by Horace Judd and Roy S. King[2] at the Ohio State University in 1906, showed that the actual velocity of the interior of the jet from a sharp-edged orifice (called by their teacher, Professor S. W. Robinson, a "frictionless orifice") averaged 99.99 per cent of that given by the formula. Any discrepancy is, of course, due to the conversion of some energy into heat.

It should be noted that at point S, well inside the tank at the level of the orifice, the energy per pound of liquid must be the same as at J; that is, $\frac{V^2}{2\,g} = h$. This energy is due to pressure and is a form of potential energy. If we go back to point T, z feet above S and J, the pressure energy[3] will be $(h - z)$ foot-pounds per pound of liquid, and the potential energy due to elevation above SJ will be z foot-pounds per pound of liquid. Therefore the total potential energy at T (measured from SJ as a datum) will be h foot-pounds per pound of liquid irrespective of the position of T. *At all points in a liquid at rest, the potential energy per pound of liquid is the same, and equals the elevation of the free surface.* As we descend from the free surface we gain pressure at the expense of elevation.

PROBLEMS

10–1. Taking g as 32.16 ft. per sec.^{2}, find the velocity of the water jet from an absolutely frictionless orifice under a head of 25.00 ft.

Ans. 40.10 ft./sec.

10–2. Solve Prob. 10–1 for mercury under a head of 3.00 in.

Ans. 4.01 ft./sec.

[2] *Engineering News*, Vol. 56 (1906), p. 327.

[3] It must be emphasized that this statement applies to the energy being carried by the liquid and not to the intrinsic energy of the liquid itself, and that it holds only for the condition specified; namely, that the liquid is being continually added to the tank at the same rate at which it flows out. If no liquid were added and the tank allowed to empty through the orifice, the energy per pound of liquid flowing out would vary from h at the start to zero at the end, with an average of $h/2$. Therefore some teachers object to the term pressure energy, and insist on always using pressure head.

10-3. What would be the per cent change in velocity in Prob. 10-1 if g were only 32.00 ft. per sec.^{2}?

10-4. Find the velocity corresponding to a head of 5800 ft. (This is the greatest head used for power, and is in Canton Valais, Switzerland.) Express also in miles per hour. *Ans.* 611 ft./sec. = 416 miles/hr.

11. Bernoulli's Theorem.—After Torricelli, no noteworthy advance in the knowledge of the mechanics of fluids was made for nearly a century. But in 1738, Daniel Bernoulli (Swiss, 1700–

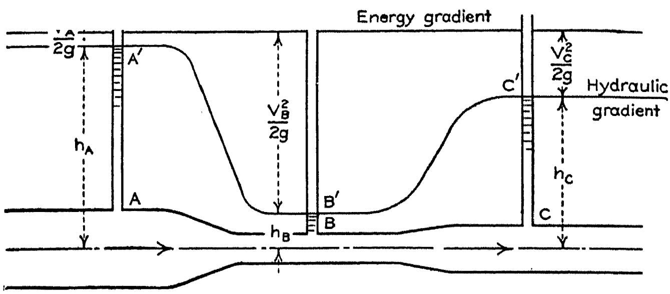


FIG. 30.—Horizontal pipe illustrating Bernoulli's theorem.

1782) announced a theorem which is still more important than Torricelli's, of which in fact Torricelli's is only a special case.

Figure 30 represents a pipe with varying diameter, in which a liquid is flowing. At points A, B, and C, piezometer tubes are inserted to show the pressure. V_A, V_B, and V_C are the mean velocities in the pipe at the respective points. At A the potential energy per pound of liquid is h_A and the kinetic energy per pound of liquid is $\frac{V_A^2}{2g}$, as shown in Art. 10. Therefore the total specific energy is $h_A + \frac{V_A^2}{2g}$. (By specific energy, we mean energy per pound of liquid.) If there is no loss of energy as the liquid flows from A through B to C, it is obvious that

$$h_A + \frac{V_A^2}{2g} = h_B + \frac{V_B^2}{2g} = h_C + \frac{V_C^2}{2g} \tag{11.1}$$

This is Bernoulli's theorem for a liquid flowing in a horizontal pipe without loss. At B the liquid has gained kinetic energy. This can only be done at the expense of pressure energy, so the

pressure at a constriction is always less than it is at the full section of a conduit. The student should impress this on his mind, as it is opposite to untrained "common sense" gained from experience in a crowd pushing through a narrow gate, which says that the pressure at a constriction is greater.

If, as in Fig. 31, the pipe is not horizontal, a change in the statement of the theorem must be made. Some line, preferably

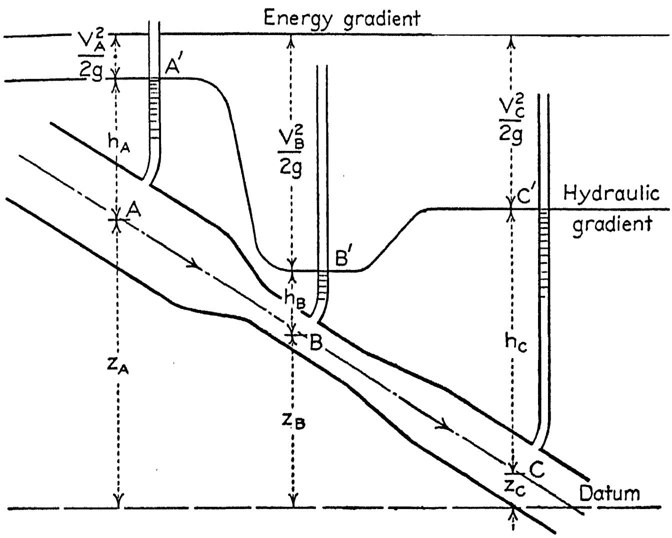


FIG. 31.—Sloping pipe illustrating Bernoulli's theorem.

as low as the lowest point under consideration, is chosen as the datum, and z is the elevation above this datum. Then at A, the potential energy due to elevation is z_A foot-pounds per pound of liquid, the potential energy due to pressure is h_A foot-pounds per pound of liquid, and the kinetic energy is $\frac{V_A^2}{2g}$ foot-pounds per pound of liquid. Then the total specific energy is

$$H_A = z_A + h_A + \frac{V_A^2}{2g} \tag{11.2}$$

This is often written

$$H = z + h_p + h_v \tag{11.3}$$

where H is called the total head, or sometimes the energy head, z is called the elevation head, or simply the elevation, $h_p = \frac{p}{w}$ is called the pressure head, and $h_v = \frac{V^2}{2\,g}$ is called the velocity head. Bernoulli's theorem simply says that the sum of these last three heads remains constant (when friction losses are not considered), and that any one can increase only at the expense of decreasing one or both of the others.

In different parts of the United States (excluding Alaska), the value of g varies from 32.12 to 32.18 feet per second per second. The value at Columbus, Ohio, is 32.16, and that will be used in this text unless otherwise stated. In slide rule work it may be considered as 32.2.

EXAMPLE I

If the data of the example in Art. 9 apply to Fig. 31, and A is 5 ft. above C and B is 3 ft. above C, and the pressure at B is 0.50 lb. per sq. in., find the pressure at A and C if the liquid is water.

0.5 lb. per sq. in. $=$ 72 lb. per sq. ft. $= \frac{72}{62.5} = 1.15$ ft. of water. At B the velocity head is $\frac{16^2}{2 \times 32.16} = 3.98$ ft. Therefore the total head is $3.00 + 1.15 + 3.98 = 8.13$ ft. This is also the total head at A and C. At A, the elevation is 5.00 and the velocity head is $\frac{4^2}{2 \times 32.16} = 0.25$ ft. Therefore, the pressure head is $8.13 - 5.00 - 0.25 = 2.88$ ft. of water $= \frac{2.88 \times 62.5}{144} = 1.25$ lb. per sq. in. It should be noted that $\frac{62.5}{144} = 0.434$ quite accurately, and as this combination occurs often, it is well to remember this figure. Its reciprocal, 2.304, the feet of water equivalent to one pound per square inch, is also useful. The corresponding values for $w = 62.4$, are 0.433 and 2.308. At C the elevation is zero, but the velocity head is $\frac{9^2}{64.32} = 1.26$, and the pressure head is $8.13 - 1.26 = 6.87$ ft. of water $= 6.87 \times 0.434 = 2.98$ lb. per sq. in.

If the pressure at B had been given in ft. of the liquid instead of lb. per sq. in., we could have found the pressure at the other points in ft. of the liquid, without knowing what the liquid was, or its density. But when the actual pressures are given, we must know the specific weight of the liquid in order to work our problem.

EXAMPLE II

A vertical 6-in. pipe contains a constriction which is 3 in. in diameter. At the constriction the absolute pressure of the flowing water is just one atmosphere, but at a point 10 ft. lower it is two atmospheres. What is the flow in the pipe (in c.f.s.), what is the velocity in the 6-in. section and in the 3-in. section, and what is the pressure 10 ft. above the constriction?

At first sight it seems that we do not have enough data, but the power of Bernoulli's theorem is illustrated by its ability to work such problems. Draw a good sketch and take your datum at 10 ft. below the constriction. Call the velocity in the pipe V, then at the constriction where the diameter is only half as much, the area will be only one-fourth as much and the velocity four times as much, or $4\ V$. One atmosphere is 33.9 ft. of water and two atmospheres $= 67.8$ ft. Equating the total heads at the constriction and 10 ft. below it, we have

$$10 + 33.9 + \frac{(4\ V)^2}{2\ g} \quad 67.8 + \frac{V^2}{2\ g} \qquad\qquad \frac{15\ V^2}{2\ g} \quad 23.9 \qquad \text{and}$$

$$\frac{V^2}{2\ g} \quad \frac{23.9}{15} \quad 1.593 \text{ ft.} \qquad \text{and} \qquad V \quad 8.02\sqrt{1.593} = 10.12 \text{ ft. per sec.}$$

Then the velocity in the constriction is $4 \times 10.12 = 40.5$ ft. per sec. The pressure 10 ft. above the constriction is 20 ft. of water less than the pressure at the lower point. $67.8 - 20.0 = 47.8$ ft. of water. $\frac{47.8}{33.9} = 1.41$ atmospheres

$$20.7 \text{ lb. per sq. in.} \quad Q = A\,V \quad \frac{}{16} \times 10.12 \quad 1.99 \text{ c.f.s.}$$

PROBLEMS

11–1. Check the flow in Example II by the velocity in the constriction times its area.

11–2. If the data of the example of Art. 9 applies to Fig. 30, find h_A and h_C if $h_B = 1$ ft. of the liquid. *Ans.* 4.73 ft.; 3.72 ft.

11–3. A horizontal pipe contracts gradually from 6 in. to 4 in. in diameter, and then enlarges gradually to 8 in. The velocity in the 6-in. pipe is 12 ft. per second and the pressure is 6 ft. of water. Assuming no energy loss, find the pressure in each of the others. *Ans.* -3.09 ft. and 7.53 ft.

11–4. An 8-in. horizontal pipe contracts gradually to 4 in. The pressure in the 8-in. pipe is 10 lb. per sq. in. and the velocity is 8 ft. per sec. Find the pressure in the 4-in. pipe if the liquid is oil weighing 50 lb. per cu. ft. *Ans.* 4.82 lb./sq. in.

11–5. A 6-in. horizontal pipe converges to 3-in. diameter. The pressure in the 6-in. pipe is 16 ft. of the liquid flowing, and the pressure in the 3-in. pipe is 4 ft. of the liquid flowing. Find the velocity in each pipe and the discharge per sec. *Ans.* 7.17 ft./sec.; 28.7 ft./sec.; 1.41 c.f.s.

11–6. By applying Bernoulli's theorem to the flow from MN to J in Fig. 29, derive Torricelli's theorem. (Assume no velocity at MN. If the tank were only 10 times as large in diameter as the jet, its area would be 100 times as

great, the velocity only 1/100 of the velocity of the jet, and its velocity head only 1/10,000 that at J. Ordinarily the tank would be larger and the error still less.)

11–7. A vertical 6-in. pipe contains a gradual constriction 2 in. in diameter. There is a pressure gage at the constriction and another in the 6-in. pipe 5 ft. higher. Each gage reads 20 lb. per sq. in. Neglecting losses, what is the velocity in the 6-in. pipe? Does it make any difference what the liquid is, or the direction in which it is flowing? *Ans.* 2.00 ft./sec.

12. Gradients, Losses, and Applications.—In Figs. 30 and 31 a line is drawn through the points $A'B'C'$. This is called the *hydraulic gradient*, and indicates the imaginary free surface, or height to which the liquid would rise in a piezometer at that point. Also drawn in these figures is the *energy gradient*, a line at every point h_v above the hydraulic gradient, and representing the total head. Since these figures represent flow without loss of energy, it is horizontal. But in the actual case there is always some *loss*, that is, some energy is converted into heat; therefore, the energy gradient always slopes downward in the direction of the flow. At a point of sudden loss of energy, as in passing through a partly closed valve, the energy gradient will take a sudden dip. But the energy gradient will never *rise* except where energy is added from an external source, as by a pump. In some problems the losses are so small that fairly accurate results can be obtained by neglecting them. In other cases losses represent, say, from 5 to 10 per cent of the available energy, and while they can hardly be neglected entirely, any method of estimating them that is not over 10 per cent in error will not introduce an error of more than 1 per cent into the final result. But in other cases energy losses represent the most important factor in the problem and they must be determined accurately. Chapters IV to VII will be concerned largely with methods of estimating losses in various cases.

It is obvious that the total head at any point in a pipe or channel must exceed the total head at some point downstream by the losses between the two points; therefore, Bernoulli's theorem with the losses considered is

$$H_1 = H_2 + h_f \tag{12.1}$$

or

$$z_1 + h_{p1} + h_{v1} = z_2 + h_{p2} + h_{v2} + h_f \tag{12.2}$$

Here the subscript 1 refers to the upstream point, and 2 to the downstream point, and h_f stands for all the losses between the two points.

As was shown in Example II of Art. 11 and in Probs. 11–5 and 11–7, Bernoulli's theorem can be used to compute the flow in a pipe if we measure the pressure at a constriction and at a point in the unconstricted pipe. In fact we do not need to know the two pressures. All we need to know is their difference. This idea was developed into a practical measuring device in 1886 by Clemens Herschel (American, 1842–1930). He called it the Venturi meter, in honor of Giovanni Battista Venturi (Italian, 1746–1822) who in 1791, had discussed the phenomenon of pressure reduction at a constriction, but had not thought of putting it to practical use. The typical form of Venturi meter is shown in Fig. 32. There is very little loss of energy in the contraction, but more in the expansion. Therefore, the pressure in the pipe should be measured upstream from the throat. The loss in the contraction seems to depend only slightly on the form of the contraction. In American practice the contraction is conical with the elements sloping about 1 in 4 or 5. In the standard German form it is much more abrupt. The shape of the expansion has little or no effect on the accuracy of the meter, but in order to cut down the total loss in the meter, it is made a smooth cone, with the elements sloping at from 1 to 10 to 1 to 20 with the axis.

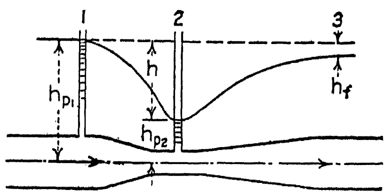


FIG. 32.—Venturi meter.

By applying (12.2) between points 1 and 2 in Fig. 32, and calling the drop in the hydraulic gradient between the two points $h = (z_1 + h_{p1}) - (z_2 + h_{p2})$, the ratio of the throat diameter to the pipe diameter r, and letting the loss $h_f = k\,h$, we find that

$$= \sqrt{\frac{2\,g\,h\,(1-k)}{1-r^4}} \tag{12.3}$$

The steps are left for the student—see Prob. 12–6. (It should be noted that this h_f is only the loss between 1 and 2, and is not the h_f shown in Fig. 32.) Usually the $(1-k)$ is removed from

under the radical and $\sqrt{1 - k}$ replaced by C, which is called the coefficient of discharge of the meter. Then the flow through the meter is given by

$$Q = \frac{C\,A_2}{\sqrt{1 - r^4}}\sqrt{2\,g\,h} \tag{12.4}$$

For mobile liquids like water, gasoline, and mercury, and at high velocities, C for $r = 0.5$ is about 0.98 (as low as 0.97 for a ¼-inch throat, and as high as 0.99 for a 100-inch throat) and is independent of the velocity. Making r less increases the coefficient slightly, and increasing r to 0.75 decreases C very appreciably. With abnormally low velocities, or with viscous liquids like oil, C may fall to 0.95 or less, and depends on the velocity. The details of these variations are discussed in Art. 56.

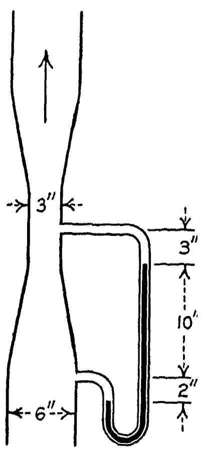


Fig. 33.

If $C = 0.98$, $1 - k = 0.96$, and the head lost between 1 and 2 is 4 per cent of the differential head. The total loss between 1 and 3 (h_f in Fig. 32), is generally between 10 and 20 per cent of h, so that the loss in expansion is from one and a half to four times as much as that in contraction.

If the meter is placed in a horizontal pipe, h is simply the difference of pressure at the two points converted into feet of the liquid flowing. When the pipe is not horizontal, this is no longer true unless the pressures are referred to the same datum. But if the pressure difference is measured by a differential manometer this is done automatically. See Fig. 33, Example II, and Probs. 12–4 and 12–5.

The Venturi meter is only one of the many applications of Bernoulli's theorem. Many others will be made in later chapters.

EXAMPLE I

The throat of a horizontal Venturi meter is 4 in. in diameter and the pipe is 12 in. in diameter. The pressure in the pipe is 20 lb. per sq. in. and the vacuum in the throat is 15 in. of mercury. The liquid flowing is water, and 4 per cent of the differential head is lost between the two gages. Find the velocity in pipe and throat, and the flow in c.f.s.

20 lb. per sq. in. = 20 × 2.304 = 46.1 ft. of water. 15 in. of mercury = 1.25 × 13.6 = 17.0 ft. of water. Therefore, h = 46.1 + 17.0 = 63.1 ft. Then h_f = 0.04 × 63.1 = 2.5 ft. Bernoulli's theorem (12.2) says that the gain in velocity head equals the loss in pressure head minus h_f, or $h_{v2} - h_{v1}$ = 63.1 − 2.5 = 60.6 ft. But since the throat diameter is 1/3 of the pipe diameter, the throat area is 1/9 the pipe area, and $V_2 = 9\ V_1$. Therefore, $\frac{V_2^2}{2\,g} - \frac{V_1^2}{2\,g} = \frac{81\ V_1^2 - V_1^2}{2\,g}$ and $V_1^2 = \frac{64.32 \times 60.6}{80}$ 48.7. V_1 = 6.98 ft. per sec., V_2 = 62.8 ft. per sec., and Q = 0.7854 × 6.98 = 5.48 c.f.s.

The negative pressure at the throat is used practically as a source of suction in filter pumps, etc. However in Venturi meters, pressures less than atmospheric should be avoided, and the above example does not represent a satisfactory working condition because air and vapor would be likely to collect and vitiate the readings. It is easy to set up problems in which the pressure at the throat is less than zero absolute. But, of course, this is not really possible. What happens is that when the pressure gets down to the vapor pressure of the liquid at that temperature, bubbles of vapor form, and A_1V_1 is no longer equal to A_2V_2.

EXAMPLE II

The Venturi meter in Fig. 33 is in a vertical pipe. The liquid in the pipe is water, and that in the manometer is mercury weighing 13.6 times as much as the water. If h_f = 4 per cent of h, find the velocity in the pipe.

The difference in pressure between the two ends of the mercury column is 12 in. of mercury = 163.2 in. of water. Then the difference in pressure between the two taps is 163.2 + 3 − 2 = 164.2 in. of water. The difference in elevation between these two points is 10 + 3 = 13 in. Therefore, $h = (z_1 + h_{p1}) - (z_2 + h_{p2}) = z_1 - z_2 + h_{p1} - h_{p2} = -13 + 164.2$ = 151.2 in. of water = 12.60 ft. of water. Notice that this is just 12.6 times the manometer reading, which agrees with the result of Prob. 3–5. This will be found always to be the case. (See Prob. 12–4.)

$$h_f = 0.04 \times 12.6 = 0.50 \text{ ft.}; \quad \text{therefore,} \quad \frac{V_2^2 - V_1^2}{2\,g} = 12.60 - 0.50 = 12.10$$

But $V_2 = 4\ V_1$, therefore 15 V_1^2 = 64.32 × 12.10, V_1^2 = 64.32 × 0.8067 and V_1 = 8.02 × 0.898 = 7.20 ft. per sec.

PROBLEMS

12–1. The diameter of the throat of a Venturi meter is one-third the diameter of the pipe. The pressure difference is 30 ft. of the liquid flowing. Find the velocity in the pipe if 4 per cent of the differential head is lost in

the contraction. Work from Bernoulli's theorem without using equations (12.3) or (12.4). *Ans.* 4.81 ft./sec.

12–2. Solve Prob. 12–1 if r is changed to 0.5, the pressure differential is changed to 24 in. of mercury, and the liquid flowing is water.
Ans. 10.57 ft./sec.

12–3. Solve Example II if the liquid flowing and above the mercury, is gasoline weighing 45 lb. per cu. ft. and all the other data remain the same.
Ans. 8.58 ft./sec.

12–4. By using letters instead of the 2 in., 3 in., and 10 in. dimensions in Fig. 33, show that h for a vertical Venturi meter, measured in feet of the liquid flowing, will always be $(s - 1)$ times the difference in elevation of the two dividing surfaces in the manometer, where s is the ratio of the specific weights of the two liquids.

12–5. Extend Prob. 12–4 to the case of a Venturi whose axis makes any angle with the horizontal.

12–6. Derive (12.3) and (12.4) from (12.2).

12–7. Taking C as 0.98 and g as 32.16, show that Q for a 2-in. throat in a 4-in. pipe is $0.1775 \sqrt{h}$ c.f.s. Find the corresponding expression for a 2-in. throat in a 6-in. pipe.

13. Power.—Power is the rate of doing work, or the rate at which energy is being transformed or transmitted. When a liquid flows through a pipe at a given rate and under a certain head, or when it issues from a nozzle or the end of a pipe, it has a definite power. If the sum of the elevation, pressure, and velocity heads is H feet, each pound of liquid flowing carries H foot-pounds of energy, and if the flow is W pounds per second, the energy passing any section per second is $W H$ foot-pounds. Then since one horsepower is 550 foot-pounds per second, the power measured in horsepower is

$$P = \frac{W H}{550} \tag{13.1}$$

Of course the value of P will depend upon the datum from which the elevation is measured. For water with $W = 62.5\ Q$,

$$P = \frac{Q\ H}{8.8} \tag{13.2}$$

No form of water wheel or turbine can develop all of the power of a given supply, because there are certain unavoidable losses.

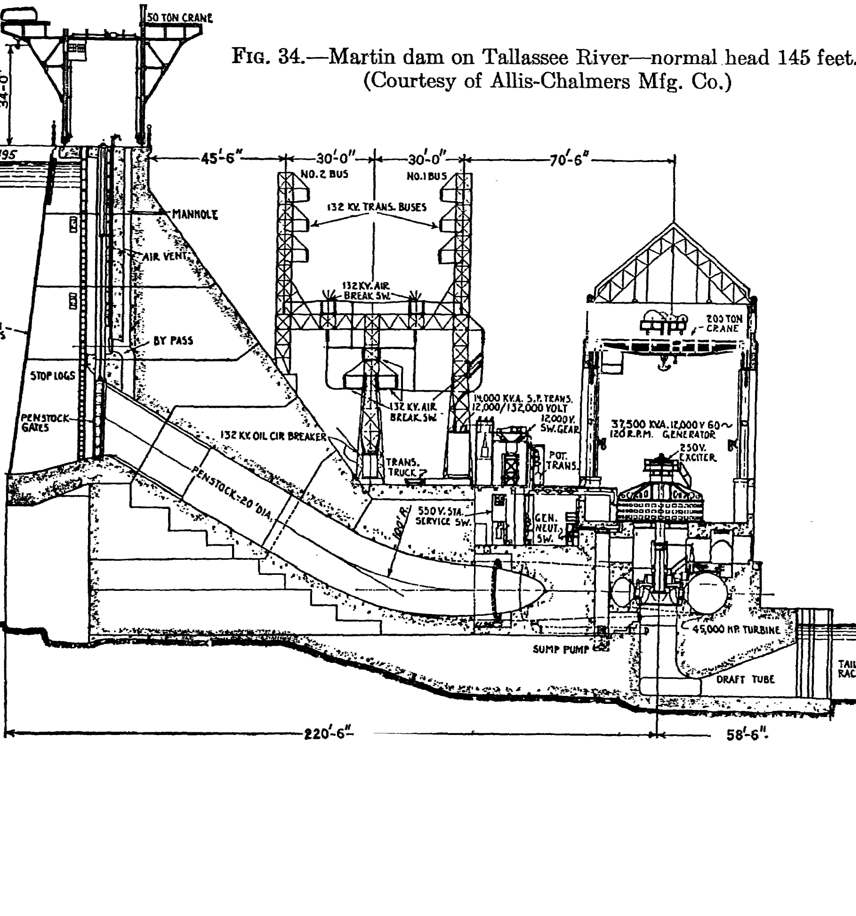


FIG. 34.—Martin dam on Tallassee River—normal head 145 feet. (Courtesy of Allis-Chalmers Mfg. Co.)

The actual power produced is that given above multiplied by the efficiency of the turbine. If it is 80 per cent, we have

$$(13.3) \qquad P = \frac{Q\,H}{11}$$

This later formula is often used as a rough rule for estimating the power which could be obtained by developing a power site on a river. If a dam is built which raises the water H feet, or if as at Niagara Falls, a drop of H feet has already been provided by nature, and if Q is the flow which can be counted on during the time that power is to be developed, formula (13.3) gives the approximate power output that can be expected.

EXAMPLE

Water is flowing in a 24-in. pipe at a velocity of 10 ft. per sec. and under a pressure of 43.4 lb. per sq. in. What is the horsepower being transmitted?

$$Q = V\,A = 10\,\pi = 31.4 \text{ c.f.s.}, \; H = \frac{43.4 \times 144}{62.5} + \frac{10^2}{2\,g} = 100 + 1.55 = 101.6,$$

$$P = \frac{31.4 \times 101.6}{8.8} = 363 \text{ hp.}$$

13–1. At 80 per cent efficiency, what power could be developed on a river with a dependable flow of 400 c.f.s., by a dam which raises the head water 44 ft. above the tail water? *Ans.* 1600 hp.

13–2. A jet 2 in. in diameter has a velocity of 100 ft. per sec. What is its horsepower? *Ans.* 38.5 hp.

13–3. Solve the above example if the velocity is increased to 20 ft. per sec., the pressure remaining the same. *Ans.* 758 hp.

13–4. If the efficiency of the nozzle is 95 per cent, and the efficiency of the pump 80 per cent, what power would be required to produce the jet of Prob. 13–2? *Ans.* 50.7 hp.

13–5. The jets issuing from the Tygart dam outlets are rectangular, 10 ft. high and 5 ft. 8 in. wide. When the jet velocity is 88 ft. per sec., what is the horsepower of one of these jets? *Ans.* 68,200 hp.

14. Limitations of Bernoulli's Theorem.—Bernoulli's theorem applies along any *stream line* [4] of a liquid at constant density and

[4] For steady, non-turbulent flow, a stream line is the same as a *path line*, the path of any one particle of the flowing liquid. In turbulent flow, it is still in the direction of the temporal average of the velocity at any point. For the distinction between path line and stream line in unsteady flow, see any book on Hydro- or Aero-dynamics, or Daugherty's *Hydraulics*, 4th Ed., McGraw-Hill Book Co. (1937), p. 63.

temperature, when acted upon by no outside forces except gravity. If there is a machine in the pipe line to add or subtract energy (a pump or a reaction turbine, for example), the head which it adds or subtracts obviously must be included as another term in the equation. Also, if heat is added to or taken from the liquid between the initial and final points, and especially if there is a change of state from liquid to vapor or the reverse, additional terms, representing the intrinsic thermal energy per pound of liquid, must be added to both members of the equation (12.2).

Engineers also apply Bernoulli's theorem to the whole body of liquid in a pipe line or channel. In most cases this introduces approximations or discrepancies which will now be enumerated.

(1) If, as is usually the case, the velocity is not the same in all parts of the cross-section, the energy head will not be the same at all points. For example, the true energy gradient for the middle of a pipe is generally higher than for the top and bottom. Engineers usually plot the energy gradient obtained by measuring up $h_p + \frac{V^2}{2g}$ from the center line of the pipe, h_p being the pressure at the center line, and V being not the velocity at the center, but the average velocity in the cross-section; that is, $\frac{Q}{A}$. The error involved is sometimes more than is realized. This matter is discussed further in Art. 39.

(2) If the flow is unsteady; that is, changing rapidly with the time, an extra term must be added to the velocity head to represent the extra kinetic energy which will exist. Also, in turbulent flow the sidewise components of the velocity involve extra kinetic energy. Since this cannot be recovered, there is some question whether it should be counted as energy, or whether the excess at the downstream point should not be included in the h_f term, just as though it were already converted into heat, as it ultimately must be.

(3) If the flow is divergent, Bernoulli's theorem does not apply strictly, but fortunately the error is small unless the divergence is quite rapid.

(4) If the stream lines are curved, the pressures are modified by the centrifugal effect, from what they would be as figured by Bernoulli's theorem applied to the whole flow. This matter is discussed further in Art. 19.

15. The Pitot Tube.—In 1730 Henri Pitot (French, 1695–1771) held a bent glass tube in the River Seine at Paris and found that water rose in the tube above the water surface a height proportional to the square of the velocity of the water. That this will be true, follows directly from Bernoulli's theorem. In Fig. 35 the liquid in the bent tube BCD is stationary, while the remainder of the water is flowing from left to right. At point A, a little upstream from the open end of the tube, the pressure head is y,

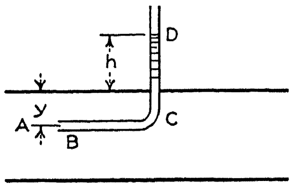


FIG. 35.—Pitot tube.

and the velocity head is $\frac{V^2}{2g}$. At the point B, just inside the tube, the pressure head is $y + h$ and the velocity head is zero. Then by Bernoulli's theorem, there being practically no loss,

$$y + \frac{V^2}{2g} = y + h \qquad \text{and}$$

(15.1) $$V = \sqrt{2\,g\,h}$$

It is more usual to write the equation

(15.2) $$V = C\sqrt{2\,g\,h}$$

This would not be necessary if the Pitot tube were always used where the flow is in parallel paths. But in turbulent flow, which is the usual case (it is described in Chapter VI), the moving particles of the liquid have sidewise as well as forward components which are constantly varying, so that at any instant the flow is at an angle with ABC. This[5] reduces C to about 0.98.

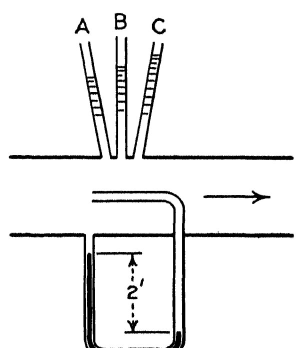


FIG. 36.

The arm BC should be placed as nearly as possible parallel with the direction of the flow, and should be long enough so that the disturbance due to CD will not extend to the mouth. In measuring the velocity in pipes flowing full under pressure, another piezometer is required to determine the

[5] See R. L. Daugherty's *Hydraulics*, 4th Ed., McGraw-Hill Book Co. (1937), pp. 168–169 and references there given.

pressure y, or a differential manometer may be used as in Fig. 36. This should open about opposite the end of the bent tube.

It is very necessary that piezometer connections be flush with the pipe surface, and perpendicular to it. The effect of inclining the tube is shown at the top of Fig. 36. B is correct, and C is tipped so that it has a slight Pitot effect and reads too high. A is tipped in the opposite direction and has a "negative Pitot effect," or slight suction, making the free surface too low.

PROBLEMS

15-1. A Pitot tube placed in a jet of water reads 20.4 ft. (above the top surface of the jet). What is the velocity of the jet? (In this case, C is 1.) *Ans.* 36.2 ft./sec.

15-2. In Fig. 36 the liquid in the pipe is oil, weighing 50 lb. per cu. ft. The liquid in the manometer is water. What is the velocity at the mouth of the Pitot, if $C = 0.98$? *Ans.* 5.56 ft./sec.

15-3. Some railroads have arranged to take water into the tender of a moving locomotive by having a scoop dip into a long narrow tank of water between the rails. If the water must be lifted 10 ft. into the tank on the tender, what is the minimum speed of train (in miles per hr.), at which this may be accomplished? *Ans.* 17.3 miles/hr.

16. Impulse of Jets.—In Fig. 37 a jet of liquid issues from an orifice or nozzle at the left, and strikes an immovable flat plate, perpendicular to its path. The liquid will turn through 90° and spread out in all directions, the stream lines being as shown. The central line AB stops at point B, which is called the *stagnation point*, because at that one point the liquid will be at rest. The force that the jet exerts on the plate will first be worked out for the following numerical case.

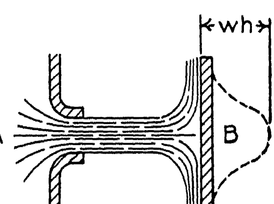


FIG. 37.

EXAMPLE I

A jet 2 in. in diameter with a velocity of 100 ft. per sec., strikes a large flat plate perpendicularly as in Fig. 37. What force does it exert, if the plate is stationary?

The area of a 2-in. circle is $\frac{\pi}{144}$ sq. ft. and Q $\frac{100\,\pi}{144}$ c.f.s. Then W, the

weight of liquid leaving the orifice per second is $\frac{62.5 \times 100\,\pi}{144} = \frac{6250\,\pi}{144}$ lb. per sec. The momentum of this mass is $\frac{W\,V}{g} = \frac{6250\,\pi \times 100}{144 \times 32.16} = 424$ lb. After the jet has been deflected through 90° by the stationary plate, it has no momentum in its original direction, so that the change in momentum is equal to the original momentum. But by Newton's second law, this change can be produced only by a force equal to the time rate of change of momentum. The change of momentum per second is 424 lb. Therefore, the force F which the plate exerts on the jet (toward the left) is 424 lb. This is also the force that the jet exerts on the plate (towards the right).

If letters instead of numbers had been used in the above example the result would have been

$$F = \frac{W\,V}{g} \tag{16.1}$$

The student should go through the proof of (16.1) preferably using a time t instead of one second. Note that the units of (16.1) check, as force has the dimension of mass times acceleration. This method of reasoning from momentum considerations, is often used in Hydraulics and the Mechanics of Fluids. It has the great advantage over energy considerations that no experimental constant is necessary. Some energy is generally lost (converted into heat), but momentum can be "killed" only by the action of a force, and *Newton's second law is exact.* **The law of continuity, Bernoulli's theorem, and the conservation of momentum, are the three major topics of this chapter, and are the fundamentals of fluid flow.**

The force which a jet can exert on a large fixed plate at right angles to its path, may be called the *impulse* of the jet (although strictly it is impulse per second). It should be noted that in producing the velocity of the jet, the tank had to exert this same force on the jet, and the jet may be thought of as pressing back on the tank with this force or "kick back." The impulse of the jet can be expressed in a different form. If the area of the jet is A, W will be $w\,A\,V$ pounds per second, and

$$F = \frac{w\,A\,V^2}{g} \tag{16.2}$$

By Art. 10, $V = \sqrt{2\,g\,h}$, where h is the head on the orifice. So $F = 2\,w\,A\,h$ and the force that the jet exerts per unit area

will be $\frac{P}{A} = 2\,w\,h$. This is just twice the unit pressure that the liquid in the tank is exerting on other areas at the same level as the orifice. This is another of the seeming paradoxes of the mechanics of liquids. But actually we have created the difficulty by introducing the idea of "force per unit area." The total force that the jet exerts is given correctly by (16.2), provided all particles of the liquid have their directions changed by 90°. The unit pressure at B, as given by writing Bernoulli's theorem from A to B, is h feet of the liquid or $w\,h$ pounds per square foot. At other points the pressure on the plate is less than at B, as shown by the dotted curve at the right of Fig. 37. But the total area over which this pressure extends, is several times as great as A, so that the total force exerted is twice as great as though the unit pressure at B acted only on area A. Roughly we can think of an average pressure of $\frac{w\,h}{3}$ acting on an area of $6\,A$. If the area of the plate is less than about this limit, the jet will not be entirely deflected and the total pressure will be less than that given by (16.2). For a small plate in a large jet (as a rectangular bridge pier in a river, or a bridge member or a building in a wind) the force will be somewhat less than half that given by (16.2), as the pressure will be $w\,h$ over most of the area, but less near the edges. But the total effect of a water current or wind on an object is another matter, for there will be a reduction of pressure on the downstream side as well as an increase on the upstream side. In rough computations the total force, including both effects, may be taken as

$$F = \frac{1.3\,w\,A\,V^2}{2\,g} \tag{16.3}$$

or 65 per cent of that given by (16.2). For more accurate work, books on Aero-dynamics should be consulted to get the proper constant to use for the particular case under consideration.

EXAMPLE II

Solve Example I if the plate is moving away from the jet at 40 ft. per sec.

In this case the jet overtakes the plate at only $100 - 40 = 60$ ft. per sec. so that the quantity of liquid striking the plate per second is $\frac{60\,\pi \times 62.5}{144}$

$= \frac{625\,\pi}{24}$ lb. per sec. The absolute velocity of the liquid after it has been deflected by the plate has a component of 40 ft. per sec. in the original direction, so that the change in momentum (mass times change in velocity) is $\frac{625\,\pi \times 60}{24 \times 32.16} = \frac{6250\,\pi}{128.6} = 152.6$ Then the force = time rate of change of momentum = 152.6 lb.

PROBLEMS

16–1. Water issues from a nozzle in a jet 3 in. in diameter at 80 ft. per sec. and strikes a large perpendicular fixed flat plate. What is the force exerted on the plate? *Ans.* 611 lb.

16–2. What would be the force if the plate were moving 40 ft. per sec. in the same direction as the jet? *Ans.* 152.6 lb.

16–3. What force would the jet of Example I exert if the plate were moving toward the jet at 60 ft. per sec.? *Ans.* 1085 lb.

16–4. A 6 ft. × 6 ft. gage house, 40 ft. high, stands in a river with one 6 ft. face perpendicular to the current. In a flood the water is 30 ft. deep and has an average velocity of 8 ft. per sec. Estimate the total unbalanced force on the gage house. *Ans.* 14,550 lb.

17. Pressure of Deflected Flow.

—The previous article covers a special case of deflected flow, but we will now consider a more general case. In Fig. 38, 160.8 pounds of liquid is striking the

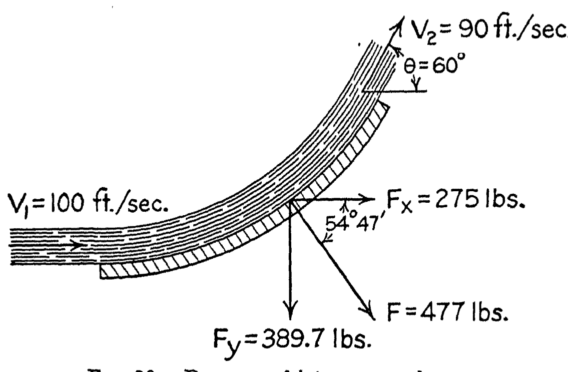


FIG. 38.—Pressure of jet on curved vane.

vane per second at a velocity of 100 feet per second and is deflected through an angle of 60 degrees.[6] Due to friction on the

[6] This assumes that the flow is "two dimensional"; that is, all in planes parallel to the paper. The vane would have to have sides, front and back, to prevent sidewise spreading of the jet.

vane, the liquid leaves the vane at only 90 feet per second. (This requires the jet to be somewhat larger when it leaves.) Then the original momentum (in the x direction) of the liquid arriving is $\frac{160.8}{32.16} \times 100 = 500$ pound-seconds per second, and the final momentum in the x direction is $\frac{160.8}{32.16} \times 90 \times \cos 60°$ $= 225$ pound-seconds per second. Then $F_x =$ change of momentum in the x direction per second $= 500 - 225 = 275$ pounds. The original y momentum was zero and the final y momentum is $5 \times 90 \times 0.866 = 389.7$ pounds. That is, the vane pushed the jet upward with this force, so the jet must have pushed down by this amount, and the resultant force of the jet on the vane is down and to the right, with the components $F_x = 275$ pounds and $F_y = 389.7$ pounds. $389.7 \div 275 = 1.417 = \tan 54° \; 47'$. $\sin 54° \; 47' = 0.8170$, and $389.7 \div 0.8170 = 477$. Therefore F is 477 pounds at $54° \; 47'$ below x.

The same result could have been obtained in another way. The jet as it leaves the vane has an impulse, as given by (16.1) of $\frac{160.8}{32.16} \times 90$ 450 pounds. It can be thought of as "kicking back" against the vane with this force. The jet struck the vane with an impulse of $5 \times 100 = 500$ pounds. The resultant of these two forces, found graphically as in Fig. 39, or by computation, is the resultant force on the vane.

The vector change of velocity may also be found as in Fig. 40. The x velocity is decreased from 100 to 45, a change of 55; and the y component is increased from 0 to $90 \times 0.866 = 77.94$.

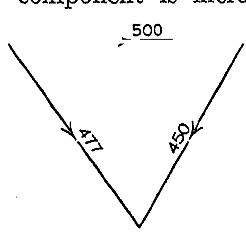


FIG. 39.

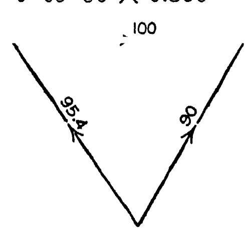


FIG. 40.

Therefore ΔV, the change in velocity $= V_2 - V_1$, is 95.4 feet per second at $35° \; 13'$ to the left of the vertical. The change in momentum is in that direction and its amount is the mass, in this

case 5, times the change in velocity, which gives 477 pounds as before.

This may be put in generalized form as follows. In Fig. 38 let V_1 and V_2 be the absolute velocities of the jet as it strikes the vane and as it leaves it, respectively. Let u = the absolute velocity of the vane in the same direction as V_1. (The case where the velocity of the vane is not in the same direction as V_1 is too complicated to consider here.) Then W, the weight of liquid striking the vane per second, is $w\,A\,(V_1 - u)$, where A is the area of the jet as it strikes the vane. F_x, the x component of the force the jet exerts on the vane, is $\frac{W}{g}$ times the decrease in the x component of the velocity (in this case, $V_1 - V_2 \cos\theta$). F_y, the y component of the force the jet exerts on the vane, is $\frac{W}{g}$ times the decrease in the y component of the velocity (in this case, $0 - V_2 \sin\theta = -V_2 \sin\theta$, the minus sign showing that it is an increase, and the force acts downward). Then the resultant force is given by the formula

$$F = \frac{W}{g}\Delta V \tag{17.1}$$

where ΔV represents the decrease in velocity; that is, the vector difference obtained by subtracting vectorially the velocity with which the liquid leaves the vane from the velocity with which it approaches it. (16.1) was simply a special case of this. ΔV can be thought of as the vector difference of the absolute velocities, or of the velocities relative to the vane. (The latter is in each case u less than the former, vectorially, and the u cancels out when the difference is taken.) In working problems numerically, it is generally better to consider the x and y components separately. This is not necessary when working graphically.

EXAMPLE I

Suppose a vane curved as in Fig. 41 so that the jet is entirely reversed in direction. Let the vane be fixed, the velocity of the jet be 100 ft. per sec., its area such that W = 160.8 lb. per sec., and friction such that the jet

leaves the vane with a velocity only 0.9 of that with which it strikes it. Find the force exerted on the vane.

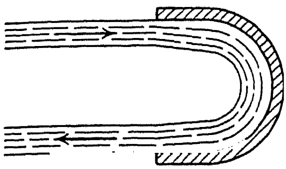


FIG. 41.

Since the velocity is changed from 100 ft. per sec. to the right to 90 ft. per sec. to the left, $\Delta V = 190$ ft. per sec., $\frac{W}{g} = \frac{160.8}{32.16} = 5$ slugs, and $F = 5 \times 190 = 950$ lb. Or the problem could have been worked by saying that the jet exerted a force of $5 \times 100 = 500$ lb. as it struck the vane and a "kick back" of $5 \times 90 = 450$ lb. as it left.

EXAMPLE II

Work Example I if the vane, instead of being stationary, is moving to the right at 40 ft. per sec. This change has two effects. The jet overtakes the vane and hits it with a velocity of only $100 - 40 = 60$ ft. per sec. Therefore only 60 per cent as much liquid will hit it as before, and $W = 0.6 \times 160.8 = 96.5$ lb. per sec. Then the jet will leave the vane with a velocity of $0.9 \times 60 = 54$ ft. per sec. Therefore the change in velocity is $60 + 54 = 114$ ft. per sec. and $F = \frac{96.5}{32.16} \times 114 = 342$ lb. Or, by using the absolute velocities, the absolute velocity of the leaving jet is $54 - 40 = 14$ ft. per sec. to the left, and the absolute velocity when it strikes the vane is 100 ft. per sec. to the right. Therefore the change in velocity is $100 + 14 = 114$ ft. per sec.

EXAMPLE III

Work Example II if the vane instead of turning the jet through 180°, turned it only 90°; that is, the relative velocity at leaving is at 90° to the relative velocity at striking the vane.

As in Example II, W is 96.5 lb. per sec., the relative velocity with which the jet strikes the vane is 60 ft. per sec., and the relative velocity with which it leaves the vane is 54 ft. per sec. But since these are at right angles their vector difference is $\sqrt{60^2 + 54^2} = 80.72$ ft. per sec. $= \Delta V$ and $F = \frac{96.5}{32.16} \times 80.72 = 242$ lb.

PROBLEMS

17-1. Work Example I if the jet were 3 in. in diameter when it strikes the vane, and the velocity 100 ft. per sec. *Ans.* 1813 lb.

17-2. Work Example II if the velocity of the vane were 20 ft. per sec. toward the jet instead of 40 ft. per sec. away from it. *Ans.* 1368 lb.

17-3. A jet of water 2 in. in diameter has a velocity of 240 ft. per sec., and strikes a vane which is moving at a velocity of 108 ft. per sec. in the

same direction, and which is so curved as entirely to reverse the direction of flow. The relative velocity with which the jet leaves the vane is 120 ft. per sec. What is the force exerted on the vane? *Ans.* 1410 lb.

17–4. Work Example II if the velocity of the vane were 90 ft. per sec. away from the jet. *Ans.* 9.5 lb.

17–5. Work Example III by finding the change in absolute velocity of the jet.

17–6. In Example III find the impulse of the jet and the "kick back" of the jet as it leaves, and their resultant (magnitude and direction).

Ans. 242 lb. at 41° 59′.

17–7. Find the magnitude and direction of the force in Prob. 17–3 if the velocity of the jet relative to the vane is changed by 165°, instead of 180°. *Ans.* 1398 lb. at 7° 08′ with original velocity.

17–8. In the preceding problem, what is the absolute velocity of the jet as it leaves the vane, and what is its cross-sectional area?

Ans. 32.1 ft./sec. at 75° 42′; 0.0898 sq. ft.

18. Work Done by Jets.—When a jet strikes a moving vane, it does work upon it equal to the distance the vane moves, multiplied by the component of the resultant force in the direction of the motion. As already mentioned in Art. 13, the rate at which work is done is called power. Obviously the power exerted on the vane cannot be more than the power of the jet, and in any actual impulse wheel or turbine it will be less than that. The ratio of the power exerted on the vanes of an impulse turbine to the power of the jet, would be the *hydraulic efficiency* of the turbine proper. Efficiencies of Pelton water wheels usually include that of the nozzle; that is, they are the ratio of the power on the vanes to the power at the beginning of the nozzle.

EXAMPLE I

Find the horsepower of the jet of Example I of Art. 17, and the power exerted on the vane in Examples I and II.

For the jet, $W = 160.8$ lb. per sec. and $H = \dfrac{100^2}{2\,g} = 155.5$ ft., therefore the horsepower is $\dfrac{160.8 \times 155.5}{550} = 45.5$. In Example I, the jet exerts a force of 950 lb., but the vane is stationary. Therefore, no work is done at all, and the horsepower of the vane is zero. In Example II, the jet exerts a force of 342 lb. and the vane moves 40 ft. per sec. in the same direction as the line of action of the force. Therefore, the horsepower is $\dfrac{342 \times 40}{550}$ $= 24.9$ hp.

The moving vanes so far studied are not practical of course. Due to air resistance and the effect of gravity, a jet can be considered as moving uniformly in a straight line for only a few feet at most, and a vane moving uniformly in a straight line would soon get out of range. But if the vanes are attached to the periphery of a wheel as in Fig. 42, before one vane gets out of range another comes in. This figure is a rough diagram of the impulse turbine or Pelton water wheel. Figure 42(b) is a view at right angles to the main view and shows how the jet is split and deflected through an angle of nearly 180°. The motion is not quite in a straight line, but it may be so considered, for approximate calculations. If the vanes are close enough together, and if they do not move too rapidly relative to the jet, all of the liquid in the jet will strike one vane or another, so that W will now be $w\ A\ V$ and not $w\ A\ (V - u)$ as in the example just worked and in Probs. 17–2 to 17–4.

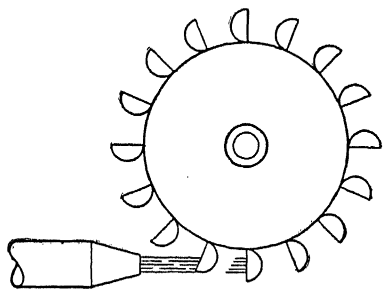


FIG. 42.—Diagram of Pelton water wheel.

EXAMPLE II

A nozzle discharges 160.8 lb. of water per sec. at a velocity of 100 ft. per sec. The jet strikes the buckets of a Pelton water wheel, which entirely reverse the direction of the jet and cause it to leave the buckets with a relative velocity nine-tenths of that with which it struck the buckets. Find the total force on the buckets and the horsepower, for several velocities of the buckets.

$\frac{W}{g}$ will now be 5 slugs. For $u = 40$ ft. per sec., $\Delta V = 60 - (-54) = 114$ ft. per sec., $F = 5 \times 114 = 570$ lb. and the power is $\frac{570 \times 40}{550} = 41.5$ hp. The results for several other velocities are tabulated below.

u	ΔV	F	Ft. Lb./Sec.	Hp.
0	190	950	0	0
20	152	760	15,200	27.6
40	114	570	22,800	41.5
50	95	475	23,750	43.2
60	76	380	22,800	41.5

If a curve of horsepower against velocity of vane is plotted, it is found that the maximum power occurs when u is 50 ft. per sec. If there had been no friction loss in the vanes, the jet would then have left the vanes with the same relative velocity as that at which it struck them (50 ft. per sec.), and ΔV would have been 100, and $F = 500$ lb. Then the work would have been 25,000 ft. lb. per sec. and the horsepower 45.5, as found in Example I.

Fig. 43.—Pelton water wheel. (Courtesy of Allis-Chalmers Mfg. Co.)

PROBLEMS

18-1. Check the line of the table given in Example II for $u = 20$ ft. per sec.

18-2. Do the same for $u = 50$ ft. per sec. and $u = 60$ ft. per sec.

18-3. By using u for the velocity of the vane, derive the general expression for horsepower in Example II.

Ans. $\dfrac{5\,(100 - u)\,1.9\,u}{550} = \dfrac{1.9\,u\,(100 - u)}{110}.$

18-4. Show that in general the power developed by a wheel with vanes that turn the jet through 180° and reduce its relative velocity from $(V - u)$ to $k\,(V - u)$, is $\dfrac{W\,(1 + k)\,u\,(V - u)}{550\,g}$.

18-5. By differentiating the answer of Prob. 18-4 with respect to u, show that the maximum power is obtained when $u = 0.5\,V$. What is its value?

18-6. What efficiency is indicated by the result of the preceding problem? (Actually, due to items here neglected, the best speed is somewhat less than $0.5\,V$, and the efficiency is less than that just found.)

18-7. Plot the results of Example II using velocity of the vanes as abscissa.

As ordinates, use successively (a) the absolute velocity with which the jet leaves the vanes, (b) the force exerted on the vanes, (c) the power.

18-8. A jet with a velocity of 100 ft. per sec. and a discharge of 160.8 lb. per sec., strikes the buckets of a Pelton water wheel which are moving at 50 ft. per sec. The relative velocity with which the jet leaves the buckets is nine-tenths of that with which it enters, and at an angle of 165° with the direction of approach. Find the total tangential force on the buckets and the horsepower. *Ans.* 467 lb.; 42.5 hp.

18-9. Show that if a flat plate just the size of a jet is held in its path, the jet will deflect a little less than 60° in all directions in the form of a hollow cone, as shown in Fig. 44. (This has actually been observed to be the case.)[7]

HINT. The unit pressure on the plate will be nearly uniformly equal to $\frac{w\,V^2}{2\,g}$ as stated in Art. 16, and the velocity of the deflected jet will be nearly the same as that of the approaching jet.

19. Curved Stream Lines.—In Fig. 45 consider the flow between the curved stream lines MN and PQ (which divide this

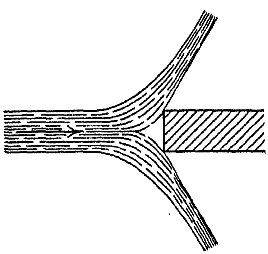


FIG. 44.

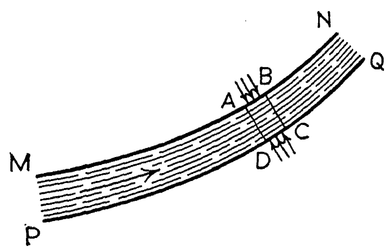


FIG. 45.

particular part of the flow from the rest of the liquid) and which are an infinitesimal distance apart. Then consider a portion of this liquid $ABCD$ of infinitesimal length. Let the radius of curvature of the curved path at this point be r, then $AD = dr$. Let the area $AB = dA$. The area CD will differ from this by only an infinitesimal of a higher order, so that the volume of the element may be taken as $dr\,dA$ and its mass as $dm = \rho\,dr\,dA$. From Dynamics we know that in order for this element to follow a curved path there must be an unbalanced force toward the center of curvature equal to $dm\,V_t^2$ where V_t is the tangential

[7] "Pitot Tubes; with Experimental Determination of the Form and Velocity of Jets," James E. Boyd and Horace Judd, *Eng. News*, Vol. 51 (Mar., 1904), pp. 316–320.

component of the velocity. Then if p is the unit pressure on the side AB, and $p + dp$ is the unit pressure on the side CD, we have $(p + dp)\, dA - p\, dA = \dfrac{dm\, V_t^2}{r} = \dfrac{\rho\, dA\, dr\, V_t^2}{r}$ and

$$dp = \frac{\rho V_t^2\, dr}{r} \tag{19.1}$$

This shows that when the stream lines are straight and r is infinite, $dp = 0$, and the pressure across the stream lines is uniform. In the Pitot tube, therefore, we could measure the pressure at the pipe wall instead of at the point inside the pipe where we were measuring the velocity. (Of course this neglects the weight of the liquid. Actually the pressure increases as we go down, the same in a flowing as in a stationary liquid.)

Where the stream lines are curved, equation (19.1) shows that the pressure is greater on the convex, and less on the concave side. The actual difference can be figured only when we know how V_t varies with r. Two cases will be considered here.

(a) *Forced Vortex.* When a vertical cylinder full of a liquid revolves about its axis and the liquid revolves with it, we have what is called a forced vortex. Here $V_t = \omega\, r$, where ω is the speed of rotation in radians per second. Then $dp = \rho\, \omega^2 r\, dr$, and by integrating between r_1 and r_2, $p_2 - p_1 = \dfrac{\rho\, \omega^2 (r_2^2 - r_1^2)}{2}$

$= \dfrac{w}{2\,g}(V_{t2}^2 - V_{t1}^2) = w\,(h_{v2} - h_{v1})$. Then

$$h_{p2} - h_{p1} = h_{v2} - h_{v1} \tag{19.2}$$

Here the subscripts 1 and 2 stand for two points in the same horizontal plane, 2 being farther from the axis than 1, V_t is the tangential velocity of rotation, and h_v is the velocity head corresponding to that velocity. If the difference of elevation of two points having the same pressure is required

$$z_2 - z_1 = h_{v2} - h_{v1}. \tag{19.3}$$

EXAMPLE I

Figure 46 represents a vertical cylinder 4 ft. in diameter and 5 ft. high, revolving on its vertical axis at 60 r.p.m. It contains a liquid which revolves with the cylinder and takes the form shown in the figure. How deep will the liquid be at the center?

ω = 1 revolution per sec. = 2 π radians per sec.; therefore, the velocity at the periphery where $r = 2$, is 4 π ft. per sec. and the velocity head is $\frac{16\,\pi^2}{64.32} = \frac{9.87}{4.02} = 2.46$ ft. At the center the velocity and velocity head are both zero. At the outside at the bottom, the pressure head is 5 ft., therefore the pressure head at the center (equals the depth of water) $= 5 - 2.46 = 2.54$ ft. Or, measuring from the bottom as datum, at the outside $z_2 = 5$ and at the center $z_1 = 5 - 2.46 = 2.54$ ft.

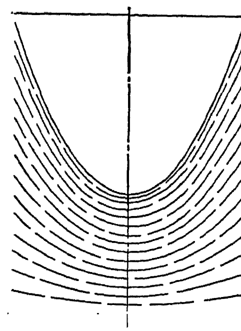


FIG. 46.—Forced vortex.

It should be noted that the change in pressure head is equal to the change in velocity head, just as in Bernoulli's theorem. Here, however, the greater the velocity, the greater the pressure head; while in Bernoulli's theorem the greater the velocity, the less the pressure head.

PROBLEMS

19-1. In the above example what would be the r.p.m. required to make the depth at the center zero? *Ans.* 85.6 r.p.m.

19-2. If the cylinder rotated at 120 r.p.m., how much of the bottom would be exposed? HINT. Imagine the cylinder to be extended downward, and find where the water surface cuts the original bottom.

Ans. A circle with diameter = 2.80 ft.

(b) *Free Vortex.* Vortices which form naturally, as on the surface of a river or when a bathtub is being emptied, are quite different from those discussed above. Here the principle of the conservation of angular momentum in the absence of external torque comes into play. By angular momentum (also called moment of momentum) we mean the product of the momentum of a particle by its distance from the axis of rotation. If a portion of the liquid moves nearer the axis, it must rotate faster so that $r\,V_t$ will remain constant. (In the case of emptying a tub, the vortex motion is superimposed on inward and downward flow.) Substituting $V_t = \frac{C}{r}$ in (19.1) gives $dp = \rho\, C^2\, r^{-3}\, dr$, and inte-

grating, $p_2 - p_1 = \frac{\rho\, C^2}{2}(r_1^{-2} - r_2^{-2})$. Therefore,

$$p_2 - p_1 = \frac{w}{2\,g}\,(V_{t1}^2 - V_{t2}^2) \text{ and}$$

$$h_{p2} - h_{p1} = h_{v1} - h_{v2} \tag{19.4}$$

This happens to be of exactly the same form as Bernoulli's theorem. If the difference of elevation of two points having the same pressure is required,

$$z_2 - z_1 = h_{v1} - h_{v2} \tag{19.5}$$

Notice that the final result of (a) and (b) can be stated in the following simple form. Any point on the free surface of a forced vortex is h_v above the lowest point on the free surface, and any point on the free surface of a free vortex is h_v below the highest point on the free surface. In each case $h_v = \frac{V_t^2}{2\,g}$, where V_t is the tangential component of the surface velocity.

EXAMPLE II

Water escaping from an opening as in Fig. 47, forms a free vortex. If a point on the free surface one in. from the axis has a tangential velocity of 8 ft. per sec., what will be the velocity at 2 ft. from the axis, and what will be the difference in elevation between the two points?

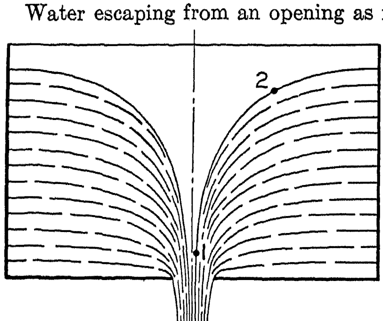


FIG. 47.—Free vortex.

The second point is 24 times as far from the axis as the first, so its velocity will be $\frac{8}{24} = 0.333$ ft. per sec. $h_{v1} = \frac{8^2}{64.32} = 0.995$ ft. and $h_{v2} = \frac{0.333^2}{64.32} = 0.0017$ ft. Therefore the second point is 0.993 ft. higher than the first.

PROBLEMS

19–3. In a free vortex without radial flow, the tangential velocity at 3 ft. from the center is 2 ft. per sec. What will it be at 1 ft. from the center, and how much lower will the free surface be there than at 3 ft. from the center? *Ans.* 6 ft./sec.; 6 in.

19-4. A free vortex in a turbulent river has a tangential velocity of 1 ft. per sec. at 2 ft. from the center. What is the difference in surface elevation between a point 5 ft. from the center and one at 1 ft. from the center?

Ans. 0.0597 ft.

Since we cannot have infinite velocity, the center of a free vortex must be hollow, as in the case of water escaping from a tub, or the central core must be another kind of flow, probably a forced vortex. The combination of a free and forced vortex is shown in section in Fig. 48. It is called the Rankine combined vortex. The part inside P and P' is a forced vortex with a constant angular velocity, the part outside is a free vortex with the velocity inversely proportional to the distance from the center. In terms of the linear velocity at P, how far is P above O? How far is P below the energy gradient MM'?

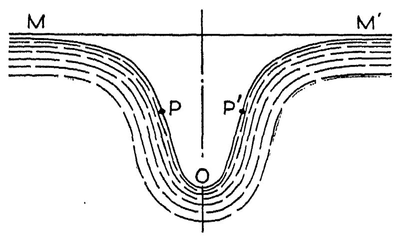


FIG. 48.—Rankine combined vortex.

PROBLEMS

19-5. An eddy in a river is a Rankine combined vortex 6 in. deep (distance of O below MM'), and has its maximum tangential velocity at 6 in. from its center. What is this velocity and what is the velocity at 5 ft. from its center? How many r.p.m. does the core make?

Ans. 4.01 ft./sec.; 0.401 ft./sec.; 76.6 r.p.m.

19-6. Plot the surface profile on one side of the center line for the preceding problem, to a scale of 1″ = 1″ vertically and 1″ = 1′ horizontally.

CHAPTER III

ORIFICES, TUBES, NOZZLES, AND WEIRS

20. Orifice Coefficients.—When a sharp-edged opening is made in the side of a tank containing a liquid as in Fig. 29, the jet which comes out is not as large as the opening. The reason for this is shown in Fig. 49. Some of the liquid approaches the orifice from the sides and, due to its inertia, cannot turn the corner at the sharp edge instantly, but has to move in a curved path. The boundary of the jet as plotted in Fig. 49 is drawn to scale from actual measurements. The orifice was 1.9975 inches in diameter, and 1.50 inches downstream from the face of the orifice plate the diameter of the jet was 1.5508 inches, or 0.7765 of the diameter of the orifice. Then the area of the jet was 0.6028 of the area of the orifice. This latter figure is called the *coefficient of contraction* and is represented by C_c. That is,

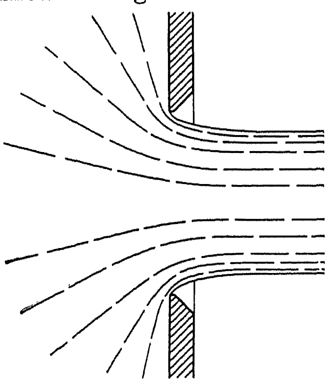


FIG. 49.

$$A_j = C_c A_o, \tag{20.1}$$

where A_j is the area of the jet at the contracted vein (section where the jet has reached its minimum size, also called *vena contracta*), and A_o is the area of the orifice. For sharp-edged orifices an inch or more in diameter, liquids as fluid as water, and with heads of 2 feet or more, this coefficient may be taken as 0.60. Its value for a more viscous liquid, or for lower heads or smaller orifices, is higher. The variation is discussed in Art. 55. Anything which hinders the liquid from approaching from the sides will also make this coefficient larger. (Even roughening the back side of the orifice plate will have a measurable effect.) Any dulling

or rounding of the edge of the orifice will also increase the coefficient. (See Art. 22.)

The velocity in the contracted vein is found to be practically uniform across the section. In the light of Arts. 10 and 11 it is obvious that

$$V = C_v \sqrt{2\,g\,H} \tag{20.2}$$

where H is the total "head on the orifice"; that is, the height of the real or imaginary free surface in the tank above the center of the orifice, plus the head corresponding to the *velocity of approach*. In a large tank this latter is negligible. For a truly sharp-edged orifice and for liquids as fluid as water, C_v can be taken as unity.[1] Worn or imperfectly made "sharp-edged" orifices may have values of C_v of 0.995 or lower. If C_v is 0.995, the kinetic energy of the jet will be 99 per cent of the original energy, the *efficiency* is 99 per cent, and the "*coefficient of loss*" is 1 per cent. These are defined as follows:

$$\text{Efficiency} = e = \frac{\frac{V^2}{2\,g}}{H} = \frac{(C_v \sqrt{2\,g\,H})^2}{2\,g\,H} \quad C_v^2 \tag{20.3}$$

$$\text{"Coefficient of loss"} = k = 1 - e = 1 - C_v^2 \tag{20.4}$$

The discharge from the orifice is

$$Q = A_j\,V = C_c\,A_o\,C_v\,\sqrt{2\,g\,H} \tag{20.5}$$

This is generally abbreviated to

$$Q = C_d\,A_o\,\sqrt{2\,g\,H} \tag{20.6}$$

where C_d is called the *coefficient of discharge*, and is equal to $C_c\,C_v$. The product $A_o\,\sqrt{2\,g\,H}$ is called the *ideal discharge*. It is what the

[1] See Note 2 of Chapter II, Art. 10. The data for Fig. 49 and Probs. 20–1 and 20–2 are from this paper. The value 0.9999 is based on Pitot tube measurements in the jets. The average value of C_v found from the measured jet diameter and the measured discharge was 0.9998. The difference may be due to air resistance slowing up the outer surface of the jets, or it may be purely experimental. "Four figure" accuracy in hydraulic measurements is too much to expect.

discharge would be if the jet came out the full size of the orifice and at the ideal velocity corresponding to the total head. C_d is usually found by measuring the actual discharge under a given head and dividing it by the ideal discharge. In the absence of actual measurements (calibration), within the limits stated above, C_d may be taken as $0.60 \times 0.9999 = 0.60$. Other cases are discussed in Art. 55.

If instead of discharging into the open air, the jet issues into another tank in which the liquid surface is above the orifice, the coefficients are practically the same, but H is now the difference in level of the two free surfaces. This is called a *submerged orifice.*

EXAMPLE

In the cap on the end of a 3-in. pipe there is a frictionless orifice 1 in. in diameter. A gage just upstream from the orifice indicates a head of 25 ft. of water. Calculate the discharge, first neglecting the velocity of approach and then considering it.

The velocity corresponding to 25 ft. of head is $\sqrt{2\,g \times 25} = 8.02 \times 5 = 40.1$ ft. per sec. The area of the orifice is $\frac{\pi}{576} = 0.00545$ sq. ft. Therefore the area of the jet $= 0.60 \times 0.00545 = 0.00327$ sq. ft. and the discharge $= 40.1 \times 0.00327 = 0.1311$ c.f.s.

Considering the velocity of approach, we have the area of the pipe = 9 times the area of the orifice $= \frac{9}{0.6} = 15$ times the area of the jet and $\frac{V_a^2}{2\,g} = \frac{V_j^2}{225 \times 2\,g}$. Then $25 + \frac{V_j^2}{225 \times 2\,g} = \frac{V_j^2}{2\,g}$ or $25 = \frac{224}{225}\frac{V_j^2}{2\,g}$ and $V_j = \frac{15}{\sqrt{224}} \times 40.1 = 40.19$ ft. per sec. and $Q = 0.00327 \times 40.19 = 0.1314$ c.f.s.

Therefore, unless it is desired to estimate discharges to four significant figures, the velocity of approach can be neglected when the pipe diameter is 3 times the orifice diameter. (But for smaller ratios it should be considered.) Actually, our answer 0.1314 is not good to four figures because C_d may be in error 1 per cent or more. In fact, in this case, due to the constricted approach, C_c will be more than 0.60. (Professor C. W. Harris in his *Hydraulics*, John Wiley & Sons (1936), p. 34, gives a curve which shows that C_d would probably have been about 0.61 in this case.)

PROBLEMS

20-1. In the paper referred to in Note 1, we find that an orifice with a diameter of 2.5000 in. and a head of 93.00 ft. of water, discharged 1.565 c.f.s. Find C_d. (The approach pipe was 2 ft. in diameter, so that no approach correction is necessary.) At the contracted vein the average diameter of the jet was found to be 1.9295 in. Find C_c and C_v from these data. A differential manometer containing a liquid with sp. gr. = 1.100 was attached to the supply tank and to a Pitot tube placed in the jet. It read 0.33 in. Then since the multiplying factor was ten, the velocity head of the jet was 0.033 in. less than the ideal velocity head. Compute C_v from these data.

Ans. 0.5936; 0.5957; 0.9965; 1.0000.

20-2. In the same paper a 1.025-in. orifice under a head of 4.02 ft. discharged 0.0567 c.f.s. The diameter of the jet was 0.8015 in. and the differential manometer read 0.07 in. Solve as in Prob. 20-1.

Ans. 0.6153; 0.6114; 1.006; 0.9999.

20-3. Work the example above, if the diameter of the approach pipe is reduced to 2 in. other data remaining the same. *Ans.* 0.1326 c.f.s.

20-4. Derive a general expression for the discharge from a frictionless orifice of area A_o in the end of a pipe of area $r\,A_o$ with a pressure head of h.

$$Ans. \quad Q = \frac{C_c A_o}{\sqrt{1 - \left(\frac{C_c}{r}\right)^2}} \sqrt{2gh}.$$

20-5. Check the last answer of the example by using the answer of Prob. 20-4.

20-6. The head required to produce the jets in Prob. 13-5 is 178 ft. What is the coefficient of velocity of these outlets? *Ans.* 0.823.

21. Re-entrant Mouthpiece.—If a short piece of pipe extends back into a tank, as in Fig. 50, it is called a re-entrant mouthpiece. If the pipe is thin, and if it extends into the tank far enough [2] so that the flow along the tank wall is negligible, it is called a *Borda mouthpiece,* in honor of Jean Charles Borda (French, 1733–1799), who first showed mathematically that this arrangement will act as an orifice with $C_c = 0.50$. This is proved by means of the results of Art. 16. The impulse of the jet (its "kick back") must be just equal to the static pressure on the cross-

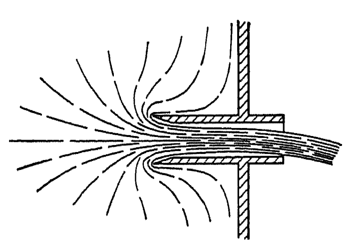


FIG. 50.—Borda mouthpiece.

[2] This is usually stated as 2.5 diameters, but Harris has shown that one diameter gives practically the same result. See his *Hydraulics,* John Wiley & Sons (1936), p. 23.

sectional area of the pipe. This must be so because the mass of water in the tank remains in equilibrium. Or we may say that the only force available to accelerate the liquid and form the jet is the relieved static pressure. By (16.2) the impulse is $\frac{w A_j V^2}{g}$. But by (20.2), $V = C_v \sqrt{2 g H}$, therefore the impulse is $2 w A_j H C_v^2$. But the static pressure on the area of the opening is $w H A_o$. Equating these two, $A_o = 2 C_v^2 A_j$, and therefore $C_c = \frac{1}{2 C_v^2}$. With $C_v = 1$, $C_c = 0.50$.

This type of orifice is rarely used for measuring flow, but is discussed here because it is the only one for which C_c can be deduced by a simple mathematical process. At first sight it might seem that the same proof would hold for the ordinary sharp-edged orifice in a flat plate, as shown in Fig. 49. But there, when the orifice is opened, not only is the pressure relieved from the area A_o, but there is also a reduction of pressure on the orifice plate in the vicinity of the orifice. Since the liquid moves along the plate toward the orifice it has some velocity head, and this can be gained only at the expense of pressure head.[3] This loss of pressure is small except near the orifice, but the total effect is enough to increase C_c from 0.50 to about 0.60.

Actually, the pipe must have some thickness and it is the outside corner that acts as the sharp edge. If the walls have a thickness of t_w and the inside diameter is D, $A_o = \frac{\pi(D + 2 t_w)^2}{4}$. As the thickness of the pipe increases there comes a time when 0.50 of this outside area is as much as 0.60 of the inside area. Then the jet begins to touch the inner corner and it acts as a sharp-edged orifice.

As seen in Prob. 21–4, for a 5-inch standard wrought-iron pipe the inside corner governs. This will be found to be true for all standard pipe smaller than this. It is also true for all "Extra Strong" pipe except 12-inch, and for all "Double Extra Strong" pipe. (This assumes the pipe cut "square" on the end. Beveling

[3] There is a similar phenomenon in the case of weirs and overflow dams which is sometimes overlooked. In fact automatic flash boards have been designed to "trip" when water flows over them to a certain depth, but have failed to do so because of this reduction in pressure.

one way or the other presents questions upon which the student may exercise his mind.)

As the re-entrant length for a thin pipe is decreased from one diameter to zero, C_c increases from 0.50 to 0.60. Definite values are given in the reference in Note 2.

The next case for which C_c was obtained theoretically, was the ideal one of a fluid of no viscosity flowing through a sharp-edged slit of infinite length. In 1869 Gustav R. Kirchhoff [4] (German, 1824–1887) showed that this would give a jet whose boundary was a tractrix, and that $C_c = \dfrac{\pi}{\pi + 2} = 0.611$. Experiments show that this applies fairly well to actual liquids as fluid as water flowing through sharp-edged orifices one or two inches high, as wide as the tank, under heads of 2 feet or more. When the width of the rectangular orifice is less than that of the tank, the coefficient is somewhat less, and may be taken as 0.60 for square as well as for round orifices.[5]

PROBLEMS

21–1. Find the ratio of thickness of pipe wall to inside diameter at which the inner corner begins to control the flow. *Ans.* 0.0477.

21–2. A 3-ft. piece of standard 12-in. W. I. pipe (outside diameter 12.75 in.) extends half its length into a large tank. Compute the discharge if the head on the mouthpiece is 36 ft. *Ans.* 21.3 c.f.s.

21–3. Solve Prob. 21–2 for a 2-ft. piece of standard 6-in. W. I. pipe (inside diameter 6.065 in., outside diameter 6.625 in.). *Ans.* 5.75 c.f.s.

21–4. Solve Prob. 21–2 for a 2-ft. piece of standard 5-in. W. I. pipe (inside diameter 5.047 in., outside diameter 5.563 in.). *Ans.* 4.01 c.f.s.

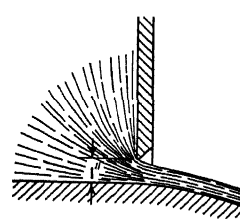


FIG. 51.

21–5. A sluice gate 5 ft. wide is raised 1 in. See Fig. 51. Estimate the flow under it if the free surface in the sluice is 9 ft. above the sill. (The jet may be assumed to be just half of that from a slit twice as wide, with the sill as its center line.) *Ans.* 6.12 c.f.s.

[4] His derivation is given in *Hydrodynamics*, by Horace Lamb, Cambridge University Press, 3rd Ed. (1906), p. 91.

[5] By means of conformal transformations, R. von Mises has worked out coefficients of contraction for a number of different shaped orifices. See *Zeitschrift VDI* (1917), pp. 447–452, 469–474, 493–498. Some further work seems to be required, however, to reconcile some of his values with experimental results.

22. Rounded Entrance.—If the space around the jet from a sharp-edged orifice were filled in as at *a* in Fig. 52, a mouthpiece would be formed from which the jet would issue the full size of the opening ($C_c = 1.00$). There would now be more drag on the surface of the jet; C_v would be about 0.98, and $C_d = 0.98$. These values are for a liquid as fluid as water, reasonably high heads, and large sizes (as described at the beginning of Art. 20) having the rounding just described. But we find that a rounding whose profile is a circular arc, as shown in Fig. 53, gives practically the same results if R is more than $\frac{D}{7}$, and if the cylindrical part of the mouthpiece is as long as D.

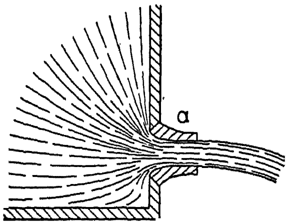


FIG. 52.—Rounded mouthpiece.

The reasonableness of this is shown as follows (following Harris). If the coefficient of contraction of a sharp-edged orifice is 0.60, the diameter of the jet is 0.7746 of the diameter of the orifice, and the diameter of the orifice is 1.291 times the diameter of the jet. Then the "contraction" on each side is 0.145 times the diameter of the jet, or approximately one-seventh.

Rounding with a radius of more than one-seventh D will give about the same coefficients as for $D/7$. Rounding with a radius of less than $D/7$ will give smaller coefficients of contraction and discharge, approaching 0.60 as the radius approaches zero.

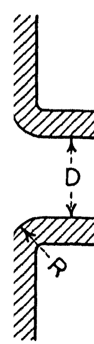


FIG. 53.

23. Nozzles.—A *nozzle* is much the same as an orifice with rounded entrance, but it is generally placed at the end of a pipe or hose so that the velocity of approach must be considered. It has many uses in addition to fire-fighting, such as in certain types of mining, in the construction of earth dams, and in the impulse type of water wheel. Figure 54 shows the ring nozzle on which John R. Freeman (American, 1855–1932) conducted his classic experiments in 1888. Smooth nozzles are so constructed that the jet is the same size as the opening and $C_c = 1.00$. A ring nozzle with an opening larger than that of a given smooth nozzle by an amount sufficient to give the same size jet, will have about

the same C_v and efficiency as the latter. C_v varies from 0.98, or more, down to 0.95, or even less in the case of very viscous liquids (see Art. 56). As pointed out in Art. 20, the efficiency is C_v^2, and

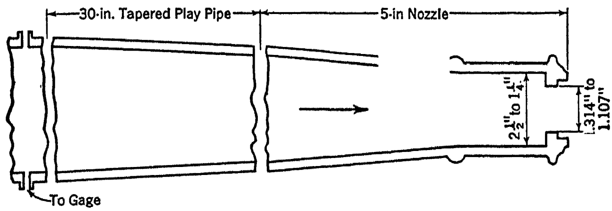


FIG. 54.—Ring nozzle used in Freeman's experiments.

will therefore generally be between 90 and 96 per cent. In problems where it is not given, take it as 95 per cent.[6]

EXAMPLE

A nozzle delivering a 2-in. jet is placed at the end of a 4-in. pipe. A pressure gage just back of the nozzle indicates a pressure head of 200 ft. Find the velocity of the jet and the discharge, assuming the efficiency as 95 per cent.

Let the velocity in the pipe be V. Then the velocity of the jet will be $4\ V$ and the velocity head of the jet will be $\frac{16\ V^2}{2\ g}$. This is 95 per cent of the total head at the beginning of the nozzle. Therefore $0.95\left(200 + \frac{V^2}{2\ g}\right) = \frac{16V^2}{2\ g}$ and $200 = \left(\frac{16}{0.95} - 1\right)\frac{V^2}{2\ g} = 15.84\,\frac{V^2}{2\ g}$, $V^2 = \frac{12{,}864}{15.84}$ 812.1 and $V = 28.5$ ft. per sec. The velocity of the jet $= 4 \times 28.5 = 114.0$ ft. per sec. and $Q = 28.5 \times \frac{\pi}{36} = 2.49$ c.f.s.

For the general case let the pressure head at the beginning of the nozzle be h_p, the mean velocity in the pipe be V_p, the velocity in the jet be V_j, and the efficiency be e. Then $e\left(h_p + \frac{V_p^2}{2g}\right) = \frac{V_j^2}{2g}$,

$$(23.1) \quad V_j = \sqrt{2\,g\,e\left(h_p + \frac{V_p^2}{2\,g}\right)} = C_v\sqrt{2\,g\left(h_p + \frac{V_p^2}{2\,g}\right)}.$$

[6] One point which is here overlooked, and which changes this result slightly, is discussed in Art. 39.

PROBLEMS

23-1. Solve the above example if the pipe diameter is changed to 6 in.
Ans. 111.2 ft./sec. and 2.43 c.f.s.

23-2. If the liquid in Prob. 23-1 is water, what is the horsepower of the jet, and how much power is lost in the nozzle? *Ans.* 53.0 hp.; 2.8 hp.

23-3. How much force would the jet of Prob. 23-2 exert on a series of vanes which reversed the direction 180° and moved at half the speed of the jet, (a) neglecting loss of velocity while in contact with the vanes, (b) assuming that the relative speed of the jet is reduced by 10 per cent while in contact with the vanes. *Ans.* (a) 524 lb.; (b) 498 lb.

23-4. What would be the combined efficiency of the nozzle and vanes in Prob. 23-3 (b)? *Ans.* 90.2 per cent.

NOTE. The actual efficiency of most Pelton wheels would not be as high as this. We have oversimplified the problem by assuming the vane always to travel parallel to the jet, and not all the power delivered to the vanes will be delivered by the wheel (because of axle friction and windage). The actual efficiency of Pelton wheels (including the nozzle) is usually between 80 and 88 per cent.

23-5. Writing the ratio of the diameter of the jet to the diameter of the approach pipe as r, develop a formula for V_j which does not contain V_p.

$$Ans.\quad V_j = \frac{C_v}{\sqrt{1 - C_v^2 r^4}}\sqrt{2\,g\,h_p}.$$

23-6. Show that $V_p = \dfrac{r^2 C_v}{\sqrt{1 - C_v^2 r^4}}\sqrt{2\,g\,h_p}$ Then use this formula to check the solution of the example.

24. Large Orifice under Low Head.—When the head on a vertical orifice is less than twice its vertical dimension, the use of the head on the center of the orifice to give the average velocity

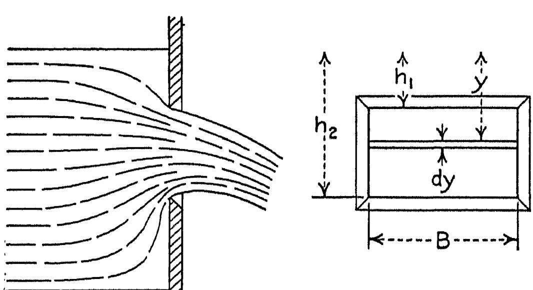


FIG. 55.—Rectangular orifice.

of the jet introduces an error. For simplicity we will consider only the case of a rectangular orifice as shown in Fig. 55.

Let the orifice be thought of as made up of horizontal strips of height dy, the head on the strip being y. Then the ideal velocity in each strip will be $\sqrt{2\,g\,y}$, and the ideal discharge, $B\,dy\,\sqrt{2\,g\,y}$. Then the total actual discharge will be

$$(24.1)\qquad Q = C_dB\,\sqrt{2\,g}\int_{h_1}^{h_2} y^{1/2}\,dy = \tfrac{2}{3}\,C_dB\,\sqrt{2\,g}\,(h_2^{3/2} - h_1^{3/2})$$

If we let H = the head on the center of the orifice $= h_2 + h_1$ and let the height of the orifice be $2\,a$, we have $h_2 = H + a$ and $h_1 = H - a$, and expending the three-halves power by the binomial theorem (24.1) reduces to $Q = \frac{2}{3}\,C_dB\,\sqrt{2\,g}\,(3\,H^{1/2}a - \frac{1}{8}\,H^{-3/2}a^3\,\ldots)$. Letting $D = 2\,a$ be the vertical dimension of the orifice, and A_o its area, this simplifies to

$$(24.2)\qquad Q = C_dA_o\,\sqrt{2\,g\,H}\left\{1 - \frac{1}{96}\left(\frac{D}{H}\right)^2\,\ldots\right.$$

Then the ratio of the true discharge to the approximate discharge as given by equation (20.6) is

$$(24.3)\qquad \frac{Q}{Q_a} = 1 - \frac{1}{96}\left(\frac{D}{H}\right)^2\,\ldots$$

If the head on the center is twice the "diameter," the error by the approximate method is only 1 part in $2^2 \times 96 = 384$, or 0.26 per cent. Since the coefficient of discharge will not be known with this accuracy, the approximate method is satisfactory. But if $H = D$, the error is 1 part in 96, or 1.04 per cent. Since H cannot be less than a, the maximum error would be 1 part in 24, or 4.17 per cent. But actually, as will be shown in Example II of the next article, the error can never be this great.

PROBLEMS

24-1. Check (24.1) by drawing your own figure and performing the integration independently.

24-2. Check (24.3), carrying to the next term of the expansion. Does this next term change the 1.04 per cent given above?

Ans. $1 - \dfrac{D^2}{96\,H^2} - \dfrac{D^4}{2048\,H^4}$; yes, to 1.09%.

25. Sharp-crested Weirs.—If the free surface in Fig. 55 is just level with the top of the orifice, the liquid will not touch the top of the opening, and we have an example of the *sharp-crested* weir shown in Fig. 56. a is called the *crest* of the weir. If it extends clear across the approach channel it is called a *full-width* weir. This is also sometimes called a *suppressed* weir, because the contractions at the sides are "suppressed." If the crest is shorter than the width of the approach channel, with sharp vertical edges limiting the flow at each side (as in the rectangular orifice), it is called a *rectangular notch* or *contracted* weir. If the sides of the notch slope, so that it is wider at the top than at the bottom, it is called a *trapezoidal notch.* A *triangular notch* has the sloping sides extended until they intersect, so that there is no horizontal crest.

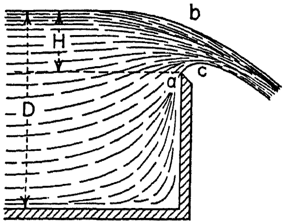


FIG. 56.—Sharp-crested weir.

The usual purpose of a weir is to measure the flow. The calculation for a full-width weir or a rectangular notch is very similar to the process used in the preceding articles. It gives

$$Q = C_d B \sqrt{2g} \int_0^H y^{1/2}\, dy = \tfrac{2}{3} C_d B \sqrt{2g}\, H^{3/2} \tag{25.1}$$

where H is the "head on the weir" as shown in Fig. 56. The student should draw a clear figure and check this equation. If we combine the $\frac{2}{3} C_d \sqrt{2g}$ into K, we have

$$Q = K\, B\, H^{3/2} \tag{25.2}$$

which is the basic formula for these weirs. It must be admitted that the derivation of (24.1) and (25.1) is unsatisfactory. If the integration is thought of as actually in the plane of the weir or orifice, the velocities are not all parallel and perpendicular to that plane, as assumed in the derivation, nor does the velocity in the discharging stream vary from top to bottom in the manner assumed. Values of C_d and K may therefore include errors of principle as well as variations due to the physical characteristics of the liquid and the orifice or weir notch. The formulas are

useful in practice, however, since experiments show that C_d and K are approximately constant for a certain range of conditions, and their variation with other values of these conditions has been very thoroughly studied experimentally.

Experiments first performed by Henry Emile Bazin (French, 1829–1917) in 1886, and since checked by many other experimenters, showed that the point in the free surface directly above the crest (b in Fig. 56), is 0.85 H above the crest. The jet from a weir is called the *nappe*. The highest point on the lower surface of the nappe (c in Fig. 56) was found by Bazin to be 0.125 H higher than the crest and about 0.26 H downstream from it.[7] See Fig. 57 for a scale drawing based on Bazin's results.

The full-width weir may be thought of as half of a rectangular orifice extending the full width of the tank, for which $C_d = 0.611$ (see last paragraph of Art. 21). Then K would be $\frac{2}{3} \times 0.611 \times 8.02 = 3.27$. (It should be pointed out that the $\frac{2}{3}$ and the C_d are dimensionless and have the same value in all systems of units, but that $\sqrt{2\,g}$ has the dimensions of length to the one half divided by time. So K has these same dimensions, and its value will be different in different systems of units.) Careful experiments made in 1852 and 1853 by James B. Francis (American, 1815–1892), showed that actually C_d is here somewhat more than 0.611, so that K for a sharp-edged full-width weir with negligible velocity of approach is 3.33. When this value is substituted in (25.2) we have

$$Q = 3.33\,B\,H^{3/2} \tag{25.3}$$

which is called Francis' formula for full-width sharp-crested weirs with negligible velocity of approach. A very complete study by Schoder and Turner[8] at Cornell University showed that this

[7] This is for negligible velocity of approach and a sharp-edged weir with a smooth upstream face. Rounding the edge, roughening the upper part of the upstream weir face, or increasing the velocity of approach (by using a lower weir), will change the 0.125 mentioned above to something less than that value. The last two may reduce it to at least 0.090, and, of course, by sufficient rounding it could be brought to zero. These changes would increase the discharge by several per cent.

[8] "Precise Weir Measurements," *Trans. A.S.C.E.*, Vol. 93 (1929) pp. 999–1190. The discussions are also valuable. See especially those by Rehbock and Lindquist, pp. 1143–1176.

formula gives the discharge of water over sharp-edged full-width weirs under heads of not less than 0.20 feet or more than one-fifth the height of the weir ($D - H$ in Fig. 56) within two per cent or

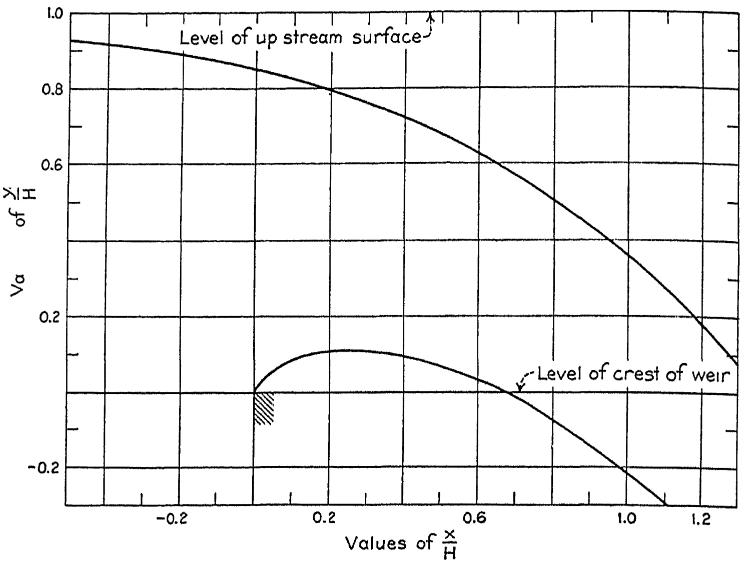


FIG. 57.—Profile of nappe from sharp-crested weir, from Bazin's data.

less. When the head is less than 0.20 feet, viscosity and surface tension have an appreciable effect. This is discussed in Art. 57. When the head is more than one-fifth the height of the weir, the velocity of approach must be considered. Formulas to correct for the effect of velocity of approach were proposed by Weisbach in 1844, by Francis in 1871, by Fteley and Stearns in 1879, by Bazin in 1888, and by Rehbock in 1912—to name only some of the better known. Rehbock's formula in our units and notation is

$$K = 3.235 + \frac{1.78}{107\,H - 1} + \frac{0.428\,H}{D - H} \tag{25.4}$$

Even this complicated formula applies only when the velocity in the approach channel is nearly uniformly distributed across the cross-section, as for example by the use of baffles. Schoder and Turner came to the conclusion that the distribution of velocity

in the approach channel is a vital factor in determining the discharge, and "the mere statement that velocities were approximately equalized by baffles leaves the matter too uncertain except for a limited range of heads and heights." They suggest a formula which requires the average velocity head of that part of the approach flow which is below the level of the crest, and that part which is above the crest level. See Prob. 25–5.

Therefore, if it is desired to measure flows with a weir to within one per cent of true values, it is necessary to calibrate the weir in the setting where it is to be used (or one exactly similar), by measuring the discharge in a given time under different heads. A *rating curve* can then be plotted of Q against H (or if preferred, of K against H). If of new iron, the weir should be recalibrated every month or so, to watch the effect of corrosion, and even a brass weir should not be used too long without rechecking. The head should be measured by a hook gage placed upstream from the crest 3 or 4 times as far as the maximum head to be used. See Fig. 58.

If the side walls of the channel do not project a short distance beyond the plane of the weir, the nappe will widen out and Harris [9] has shown that K will be increased by $0.2\frac{H}{B}$ On the other hand if the side walls do extend beyond the weir, it is very important that ample facility be supplied for air to get into the space under the nappe. Otherwise the air that was originally there will be gradually carried away; the pressure will become less than atmospheric; the nappe will be drawn down; and the discharge increased.

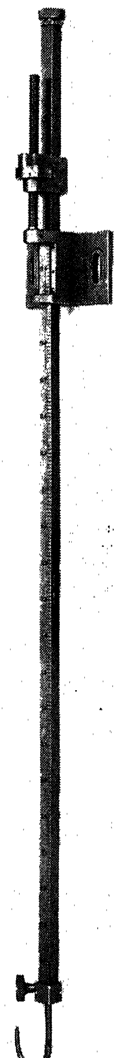


FIG. 58.—Hook gage. (Courtesy of W. & L. E. Gurley.)

[9] *Influence of Two Secondary Factors in Weir Measurements* by C. W. Harris, Bull. 81, Engineering Experiment Station, University of Washington (1935).

For a rectangular notch, where the liquid has ample opportunity to approach from the sides (the width of the approach exceeding B by at least $4\,H$), the "end contraction" will reduce K by about $0.3\,\frac{H}{B}$, and we have

$$Q = \left(3.33 - 0.3\,\frac{H}{B}\right) B\,H^{3/2}. \tag{25.5}$$

This formula becomes inaccurate, however, if H is too large compared to B. H should always be less than B, and preferably not more than one-third of B.

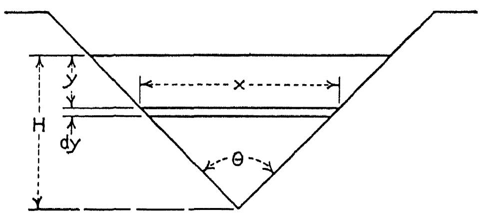


FIG. 59.—Triangular notch weir.

Figure 59 shows the end view of a triangular notch weir. Since $x = 2\,(H - y)\tan\frac{\theta}{2}$, it is easy to show by the method of Art. 24 that

$$Q = C_d \int_0^H \sqrt{2\,g\,y}\;x\,dy = \frac{8}{15}\,C_d\,\sqrt{2\,g}\,\tan\frac{\theta}{2}\,H^{2.5} \tag{25.6}$$

See Prob. 25–7. This can be simplified to

$$Q = K\,H^{2.5} \tag{25.7}$$

by putting $K = \frac{8}{15}\,C_d\,\sqrt{2\,g}\,\tan\frac{\theta}{2}$. For a 90° notch with sharp edges and $H = 0.4$ ft. or more, experiments show K to be about 2.50. Its value for lower values of H is discussed in Art. 57. The chief advantage of a triangular notch is that the same weir will measure a wide range of flows with satisfactory accuracy.

All of the statements in this article and the next assume that the liquid is water. As a matter of fact, they are also true for any other liquids which are no more viscous than water. The question of the effect of viscosity on weir flow is discussed in Art. 57.

EXAMPLE I

Run 21 of Series D of Schoder and Turner's experiments on sharp-crested, full-width weirs gave the following data. $B = 4.22$ ft., $H = 0.4536$ ft., $Q = 4.3083$ c.f.s. Find K and compare it with Francis' value.

$$0.4536^{3/2} = 0.4536\sqrt{0.4536} = 0.3055. \qquad \text{Therefore} \quad K = \frac{4.3083}{0.3055 \times 4.22}$$

$= 3.342$. This exceeds Francis' value by 0.012 or 0.36 per cent.

EXAMPLE II

Consider a rectangular orifice such as Fig. 55, 3 ft. wide and 1 ft. high, but running only partly full, that is, flowing as a notch weir. If the flow is increased, at what point will it begin to operate as an orifice, and what will the discharge be at that point?

Let z be the elevation of the free surface in the tank above the bottom of the orifice. Then it begins to act as an orifice when point b of Fig. 56 touches the top, so that $0.85\, z = 1$ and $z = 1.176$ ft. By (25.5) the weir discharge will be $\left(3.33 - \frac{0.3 \times 1.176}{3}\right) 3 \times 1.176^{3/2} = 12.29$ c.f.s. At this point the flow suddenly changes to that through an orifice. The head on the center of the orifice is $1.176 - 0.500 = 0.676$ and the approximate formula gives $Q = 0.60 \times 3 \times 8.02\sqrt{0.676} = 11.87$ c.f.s. But at such low heads we should correct this by the result of Prob. 24–2.

$$\frac{D}{H} = \frac{1.0}{0.676} = 1.480 \qquad \text{and} \qquad \frac{D^2}{96\,H^2} + \frac{D^4}{2048\,H^4} = \frac{2.190}{96} + \frac{4.796}{2048} = 0.0251$$

Therefore the above result must be reduced by 2.51 per cent, and $Q = 11.87 - 0.30 = 11.57$ c.f.s. Thus with a gradually rising head, just as the head reaches 1.176 ft. the discharge will suddenly change from 12.29 c.f.s. to 11.57 c.f.s.

PROBLEMS

25–1. By Francis' formula, estimate the discharge over a sharp-crested full-width weir, 5 ft. wide, under a head of 2.25 ft. *Ans.* 56.2 c.f.s.

25–2. Run No. 20 of Series L of Schoder and Turner's experiments on a full-width weir 4.22 ft. long, gave a discharge of 4.2958 c.f.s. under a head of 0.4532 ft. Find K. By what per cent does it differ from Francis' value?

25–3. Runs 33 to 37 of Series L of the same experiment gave a mean head of 2.2451 ft. Compute the discharge per foot of width by Francis' formula. The experiments gave 11.914 c.f.s. per ft. of width. What is the per cent error in the formula? *Ans.* 11.20 c.f.s.; 5.98%.

25–4. Find the discharge per ft. of width of weir by Rehbock's formula, for a weir 4 ft. high under a head of 1 ft. What would be the velocity of approach in this case? *Ans.* 3.359 c.f.s./ft.; 0.672 ft./sec.

25–5. Schoder and Turner's formula for sharp-edged full-width weirs may be put in the form $Q = 3.33\,B\left[(H + h_{va})^{3/2} + \frac{H\,h_{vb}}{3.33}\right]$, where h_{va} is the mean velocity head of approach above the weir crest level, and h_{vb} is the mean velocity head of approach below the weir crest level. Assuming that the velocity of approach is uniformly distributed, find by this formula the discharge per foot of width for the data of the preceding problem.

Ans. 3.372 c.f.s./ft.

25–6. According to the statement in the text, for what range of heads on a weir 4 ft. high, would Francis' simple formula give results correct to within 2 per cent? What range for a 1 ft. weir? Of course the error may be less than 2 per cent outside the range specified. What about the case of Probs. 25–2 and 25–3 and Example I, where $D - H = 7.50$ ft.?

25–7. Make a complete derivation of equation (25.6).

25–8. What value of C_d gives $K = 2.50$? *Ans.* 0.584

25–9. Mercury weighing 846 lb. per cu. ft. flows from a sharp-edged 90° notch weir under a head of 1.08 in. Estimate the discharge in lb. per min. NOTE. Since mercury is much more fluid than water, the 0.4 ft. limitation mentioned in the text may be ignored. *Ans.* 308 lb./min.

25–10. What head would be required in Prob. 25–9 to produce twice as much flow? 32 times as much flow? *Ans.* 1.425 in.; 4.32 in.

25–11. In his discussion of Schoder and Turner's paper, Mr. Erik Lindquist gives a formula for sharp-crested full-width weirs based on Schoder and Turner's series D to O, inclusive, together with some of Bazin, and some of the Swiss Hydrographic Bureau at Amsteg. In our notation it is $C_d = 0.621 + \theta\left(C_d\frac{H}{D}\right)^2 + \frac{0.0007}{H^{3/2}}$. (For meaning of D, see Fig. 56.) The 0.621 is the value for the case of negligible velocity of approach, and such a large size that viscosity would be of no effect. The second term corrects for velocity of approach and the third term for the effect of viscosity. Taking θ as 0.57, and the C_d in the approach term as 0.63, show that this equation reduces to $K = 3.320 + \left(\frac{1.1\,H}{D}\right)^2 + \frac{0.004}{H^{3/2}}$.

25–12. For $D = 7.50 + H$ as in Series D, L, and M, and $H = 1.00$ ft., find K by the formula of Prob. 25–11. *Ans.* 3.341

25–13. According to the formula of Prob. 25–11, for what value of H in Prob. 25–12 would K be a minimum, and what is this minimum value? (The calculus solution leads to an equation which must be solved by trial. It is easier to solve the original equation by trial. Let $H = 0.60$ ft. for a first trial.) *Ans.* $K = 3.335$

25–14. For Example II, plot two curves of Q as a function of z on the same axes, one for the weir, and one for the orifice. Let z vary from 0.4 to 1.4 to the scale of 1 in. = 0.1 ft. and Q from 0 to 14 to the scale of 1 in. = 2 sec. ft. Compute points for $z = 0.4$, 0.7, 1.0, and 1.2 for the weir, and 1.0, 1.2, and 1.4 for the orifice. Plot these points and the answers of Example II, and draw smooth curves through them, showing the impossible part dashed. What would be the discharge when $z = 1.1$ ft.?

25-15. In 1879 the American engineers Fteley and Stearns proposed the following formula as representing the results of their experiments and those of Francis:

$$Q = 3.31\,B\,H^{3/2}\left(1 + 1.5\,\frac{h}{H}\right)^{3/2} + 0.007\,B$$

Here h is the velocity head of the water in the approach channel. Since this is a minor part of the total, let the velocity of approach be $\dfrac{3.33\,B\,H^{3/2}}{B\,D} = 3.33\dfrac{H^{3/2}}{D}$, and expand the parenthesis to the three-halves power by the binomial theorem, neglecting all terms after the first power of $\dfrac{h}{H}$. Then show that the formula reduces to $K = 3.31 + \left(\dfrac{1.13\,H}{D}\right)^2 + \dfrac{0.007}{H^{3/2}}$ and compare with the result of Prob. 25-11.

26. Round-crested Weirs.—If the space under the lower nappe in Fig. 56 is filled in with concrete, we have a fairly good shape for the spillway of a dam, and the discharge is nearly the same as for the corresponding sharp-crested weir. But c now becomes the crest of the round-crested weir, and the head is only 7/8 as much as before. (See Bazin's result in Art. 25.) Then calling the head on c, H, Francis' formula becomes

(26.1) $$Q = 3.33\,B\,(\tfrac{8}{7}\,H)^{3/2} = 4.07\,B\,H^{3/2}$$

Actually the drag of the curved surface has a retarding effect and it is better to write

(26.2) $$Q = K\,B\,H^{3/2}$$

using $K = 4.02$ for the designed head on a spillway shaped exactly to fit the nappe from a sharp-crested weir with D much greater than H. The value $K = 4.00$ is often used.[10] For smaller values of D, Q will be somewhat greater. There is only one head on a sharp-crested weir that will make a nappe of the shape of the spillway; that is, the spillway can be designed for only one head. When the head on the spillway is less than the designed

[10] Values such as 3.80 or lower have also been used, with the idea of including a "factor of safety." This is necessary at some point in the design, but it is the writer's opinion that it should be done by increasing the designed Q, or by providing additional free board, and that the values used for coefficients should be kept close to their true values.

head, the coefficient will be less, becoming as small as 3.20 for about one-tenth the designed head. For these heads there is pressure between the nappe and the spillway. On the other hand, heads greater than the designed head give coefficients

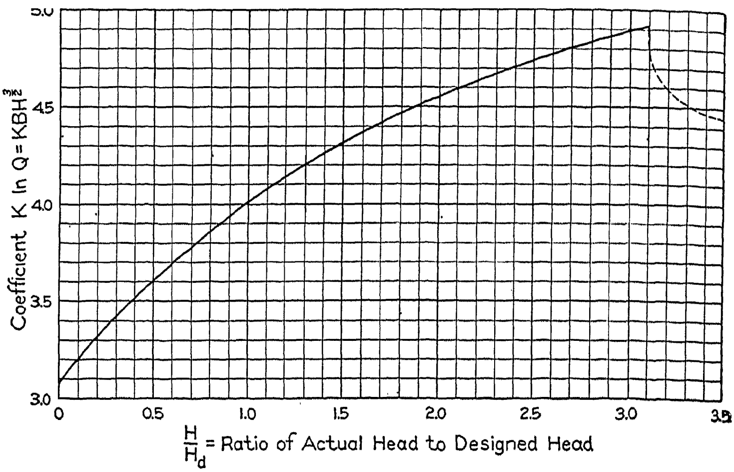


FIG. 60.—Coefficient for round-crested weirs.

greater than 4.00, three times the designed head giving about 4.90. For these higher heads, there is negative pressure between the nappe and the dam (which would have to be considered in studying its stability), and there is always the possibility that the nappe may spring clear. If it did in an intermittent fashion, the impact and vibration might destroy the dam. Therefore, the usual engineering practice is to design for the maximum head ever to be expected. Some "free board" is generally provided above this. Figure 60 shows the relation between K and H. The curve is modified slightly from that given by Dillman.[11] But even slight alterations in the shape of the crest, especially the shape from a to c, would change K.

EXAMPLE

A spillway is designed for a head of 9.00 ft. and is 200 ft. long. Estimate the discharge under heads of 9 ft. and 13.5 ft.

For $H = 9$ ft., $K = 4.00$ and $Q = 4.00 \times 200 \times 9^{3/2} = 800 \times 27 = 21{,}600$ c.f.s. For $H = 13.5$, $K = 4.33$ and $Q = 4.33 \times 200 \times 13.5^{3/2} = 866 \times 49.60 = 42{,}950$ c.f.s.

[11] For this curve and a good treatment of the whole subject, see "Model Research on Spillway Crests," by Hunter Rouse and Lincoln Reid, *Civil Engineering*, Vol. 5 (Jan., 1935) pp. 10–14.

PROBLEMS

26–1. Dover dam of the Muskingum Watershed Conservancy District has an Ogee spillway 338 ft. long and was designed for a head of 20 ft. Estimate the discharge under that head. *Ans.* 122,000 c.f.s.

26–2. Model tests on the above dam indicated that the discharge would be 40,000 c.f.s. under a head of 10.4 ft. Compute the coefficient for this condition and compare with Fig. 60. *Ans.* 3.53

26–3. Solve Prob. 26–2 for 10,000 c.f.s. for which the model indicated a head of 4.3 ft. *Ans.* 3.32

26–4. By means of Fig. 60 estimate the discharge of Dover dam under a head of 15 ft. *Ans.* 75,600 c.f.s.

26–5. A spillway is to discharge 60,000 c.f.s., and is to be carefully shaped to fit the lower nappe of a sharp-crested weir. How long should it be if the maximum head is not to exceed 9 ft.? *Ans.* 552 ft.

26–6. If in the preceding problem there is room for only a 300 ft. spillway, for what head should it be designed? *Ans.* 13.5 ft.

27. Broad-crested Weirs.—In Fig. 61, *EFGH* is a broad-crested weir (bounded by headwalls parallel to the paper and a

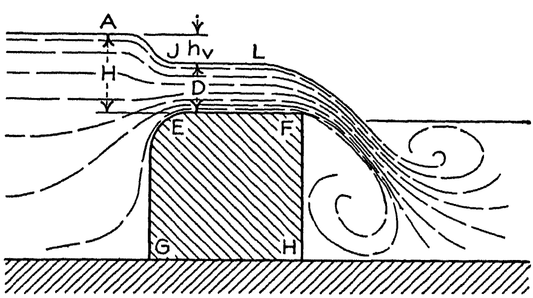


FIG. 61.—Broad-crested weir.

distance B apart). EF should have a slight downstream slope to balance the energy loss, so that JL will be parallel to EF. Call the depth of this section D. Then

$$Q = B\,D\,V \tag{27.1}$$

For the present V is assumed uniform across EJ. Writing Bernoulli's theorem from A to J gives

$$H - D = \frac{V^2}{2\,g} \qquad \text{and} \qquad V = \sqrt{2\,g\,(H - D)}$$

Therefore

$$Q = B\,D\,\sqrt{2\,g\,(H - D)} \tag{27.2}$$

This equation gives zero discharge for $D = 0$ or $D = H$, but positive discharges for all values of D between 0 and H. Q is a maximum when Q^2 is a maximum, which occurs when $(H - D)\,D^2$ is a maximum. Differentiating this with respect to D and putting the result equal to zero, $2\,D\,H = 3\,D^2$ and

$$D = \tfrac{2}{3}\,H \tag{27.3}$$

Experiments show that this is what actually takes place; that is, one-third of the available head is used in producing velocity, the velocity head $= \dfrac{H}{3} = \dfrac{D}{2}$, and

$$V = \sqrt{\tfrac{2}{3}\,g\,H} = \sqrt{g\,D} \tag{27.4}$$

Substituting this value back in (27.2) gives

$$Q = B\,\frac{2\,H}{3}\,\sqrt{\frac{2}{3}\,g\,H} = 3.09\,B\,H^{3/2} \tag{27.5}$$

The flow from J to L is spoken of as flow at *critical depth.* This idea is very important in the theory of the flow in open channels. It will be referred to again in Chapter V.

Since a spillway such as studied in Art. 26 might be expected to act as a broad-crested weir when operating under very low heads, the curve of Fig. 60 has been made to go through 3.09 for $H = 0$. It must be admitted, however, that laboratory tests on small broad-crested weirs have not agreed consistently with this value.[12]

In the last three articles it is assumed that the *tail-water* (the water surface downstream from the weir) is below the crest of the weir. When it is higher, the weir is said to be submerged. One

[12] For some of the reasons for the disagreement, see *Fluid Mechanics for Hydraulic Engineers* by Hunter Rouse, McGraw-Hill Book Co. (1938), pp. 319–326.

hundred times the ratio of height of tail-water above crest, to H, is called the *per cent of submergence.* Some experiments on the effect of partial submergence on the flow over sharp-crested full-width weirs were made by Francis in 1848, some by Fteley and Stearns in 1877, and some more by Francis in 1883. These were recomputed, tabulated, and discussed by Herschel[13] in 1885, who gave a graph and table showing the ratio of what the head would have been for the same flow if there had been no submergence to what it actually was when partially submerged plotted against the per cent of submergence. In 1928 Professor G. N. Cox[14] published new experimental data which approximately confirmed the earlier figures, but are undoubtedly more accurate. They show that a submergence of 20 per cent reduces the flow only 2 per cent, but that a submergence of 83 per cent reduces the flow 50 per cent. Values for other per cents are given in his paper. In the case of round- and broad-crested weirs, submergence may be greater without affecting the flow. No systematic study of this important subject has come to the writer's attention. We know, however, that low dams in rivers "drown out" if the flow becomes great enough. The submergence is practically 100 per cent, and the flow the same as though no dam were there.

PROBLEMS

27-1. Estimate the flow over a broad-crested weir 10 ft. long, under a head of 4 ft. *Ans.* 247 c.f.s.

27-2. Do the same for a head of 9 ft. How deep would the water be over the crest? *Ans.* 834 c.f.s.; 6 ft.

27-3. Estimate the flow over a sharp-crested weir placed across a rectangular channel 20 ft. wide if the elevation of various points with respect to some datum are: floor of channel, 100.00; crest of weir, 110.00; water surface 15 ft. upstream from weir, 114.00; water surface 15 ft. downstream from weir, 113.32. (Use answer to Prob. 25-11, and 83 per cent.) *Ans.* 274 c.f.s.

28. Discharge under Variable Head.—All of the flow so far considered has been *steady* flow; that is, it has not varied with

[13] "The Problem of the Submerged Weir," *Trans. A.S.C.E.*, Vol. 14 (1885), pp. 189–196.

[14] *Flow of Water over Submerged Sharp-crested Weirs*, G. N. Cox, Bull. 67, University of Wisconsin (1928).

time. This is an ideal which is generally only approximated under actual conditions, because all ordinary flows are accompanied by pulsations even if the average flow (say, per minute) remains fairly constant, and in city water supply and similar situations there is a continual change in the average flow. The general case of *unsteady* flow, or as it is sometimes called, nonpermanent or variable flow, cannot be considered here.[15] Only two cases will be discussed—flows from the orifice and the weir under heads which fall in some definite manner which will make the rate of flow susceptible to fairly simple mathematical analysis. A special case of pulsating flow—water hammer—will be treated in Art. 40.

If liquid flows from a tank through an orifice of area A_0 having a coefficient of discharge C_d, and if the head on the orifice is h, we have already found that $Q = C_d A_0 \sqrt{2\,g\,h}$. If A is the area of the liquid surface in the tank at a given instant, and if no liquid is supplied to the tank, the surface will fall a distance dh in time dt, such that

$$-A\,dh = Q\,dt = C_d A_0 \sqrt{2\,g\,h}\,dt \tag{28.1}$$

The negative sign is necessary because h is decreasing as t is increasing. If A is a constant, or if it can be expressed as an algebraic function of h, the equation may be integrated to give the time for h to decrease from any original value to any final value; but the results will be only approximate if the final head is too small, because, as we have seen, there is an error in applying the above orifice formula to low heads. Even at heads as large as ten times the diameter of the orifice, there is a slight error in assuming that C_d is the same as it is for heads 25 or more times the diameter. Obviously, to extend the formula to the case where $h = 0$, neglects the fact that the orifice will cease to flow full before then. But if the original head is large compared with the size of the orifice, the total time required to empty a tank through an orifice will be given approximately by integrating from the original head to zero.

[15] For a general treatment see G. K. Palsgrove and W. J. Moreland, *Variable Flow of Fluids*. Rensselaer Polytechnic Inst., Eng. and Sci. Series, Bull. 44.

EXAMPLE I

Estimate the total time required to empty a tank which is a paraboloid of revolution with every horizontal section a circle whose radius equals the square root of the height above the bottom, through a 1-in. round sharp-edged orifice in the bottom. The original depth is 9 ft.

When the head on the orifice is h ft., the area A will be πh sq. ft. Then $-\pi h\, dh = \dfrac{0.60\, \pi \sqrt{2\, g\, h}\, dt}{4 \times 144}$ and $t = -119.7 \int_9^0 h^{0.5}\, dh = 119.7 \left[\dfrac{h^{1.5}}{1.5}\right]_0^9$ = 2155 sec. or 35 min. and 55 sec.

The corresponding weir problem can be set up in a similar manner. In the case of tanks or reservoirs which are geometrical solids a relationship between A and h can usually be established which will permit of a mathematical solution. In natural reservoirs, however, the area of the liquid surface cannot be expressed as an algebraic function of the head on the weir, and the solution by means of the calculus cannot be employed. The general relation between h and t is as follows:

$$-A\, dh = Q\, dt = K\, B\, h^{1.5}\, dt \tag{28.2}$$

EXAMPLE II

A flume 60 ft. long and 10 ft. wide is discharging over a full width weir at the downstream end with a head of 12 in. If the supply is suddenly cut off, how long will it take the head to fall to 3 in.?

$$-60 \times 10\, dh = 3.33 \times 10 \times h^{1.5}\, dt, \qquad dt = -18.02\, h^{-1.5}\, dh,$$

$$36.04 \left[h^{-0.5}\right]_{1.00}^{0.25} = 36.04\,(2 - 1) = 36.0 \text{ sec.}$$

The student should be able to extend these methods to problems in which there is inflow as well as outflow, or to the case of submerged orifice flow where the tail-water level varies as well as that of the head-water. When the surface area is a function of the head on the weir which cannot be satisfactorily expressed as an equation, it will be necessary to solve the problem by a step method, assuming that the average flow between two heads not much different will be the average of the flows at the two heads. Such problems are discussed in Part VII of the Technical Reports of the Miami Conservancy District.

EXAMPLE III

A rectangular tank, 40 ft. long and 10 ft. wide, is divided by a bulkhead into two tanks, each 20 by 10, as shown in Fig. 62. One half is then filled to a depth of 10 ft., and a sharp-edged orifice 3 in. in diameter and 1 ft. from the floor is opened in the bulkhead. What time will elapse before the water level is the same in the two tanks?

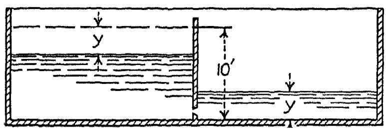


FIG. 62.

Assume $C_d = 0.60$. The time required to fall the first foot is given by integrating the equation $-200\,dh = \frac{0.60\,\pi\,\sqrt{2\,g\,h}\,dt}{64}$ from $h = 9$ to $h = 8$.

This gives $t_1 = \frac{12{,}800}{0.60\,\pi \times 8.02}\int_8^9 h^{-0.5}\,dh = 1693\,[h^{0.5}]_8^9 = 1693\,(3.000 - 2.828)$ $= 291$ sec. required to fall the first foot on the full side and rise the first foot on the empty side.

Now the orifice becomes submerged and the net head becomes $10 - 2\,y$ where y is the change in water depth on one side. If this is now called h, $2\,y = 10 - h$, and $dy = -0.5\,dh$. The equation for this part of the flow is $200\,dy = \frac{0.60\,\pi\,\sqrt{2\,g\,h}\,dt}{64} = -100\,dh$ where h varies from 8 to 0. This part of the time is $t_2 = \frac{6400}{0.60\,\pi \times 8.02}\int_0^8 h^{-0.5}\,dh = 2394$ sec. Then the total time is $291 + 2394 = 2685$ sec. $= 44$ min. and 45 sec.

PROBLEMS

28–1. A rectangular tank 10×12 ft., has a 6-in. square sharp-edged orifice in one end with its center 2 ft. above the floor. If the water is originally 11 ft. deep, how long will it take the depth to fall to 6 ft., if there is no inflow? Draw a clear sketch and show how your differential equation is set up and integrated. *Ans.* About 200 sec.

28–2. How long would it take the water surface in the above problem to fall the additional 4 ft. to the orifice? *Ans.* About 399 sec.

28–3. Show that the time required to empty a prismatic tank through an orifice in the bottom is twice the time that would have been required if the discharge had continued at the original maximum rate.

28–4. A tank is 9 ft. deep and has sloping sides, so that it is 10 ft. square on the bottom and 20 ft. square at the top. Starting full, how long will it take to empty, if there is a 1 ft. square sharp-edged orifice in the bottom? *Ans.* 233 sec.

28–5. Solve Prob. 28–1 if there is a uniform inflow into the tank of 1.203 c.f.s. *Ans.* 338 sec.

28-6. Investigate the case of Prob. 28-1 with a uniform inflow of 2.406 c.f.s.

28-7. In problems such as these, would changing the density of the liquid change the time of emptying?

28-8. Show that if successive liquid surfaces are circles with their radii proportional to the fourth root of the height above the orifice, the rate of decrease of head will be uniform. (It is said that this bi-quadratic form was practically achieved in the water clocks of ancient Egypt.)

28-9. The Leesville Reservoir of the Muskingum Conservancy District has an outlet with a cross-sectional area of 40.6 sq. ft. Assuming that this will discharge with a coefficient of 0.65, estimate the time required to reduce the head from 44 ft. to 43 ft. when there is no inflow, if the average area of the surface of the reservoir at these heads is 75 million sq. ft.

Ans. 14.9 hrs.

28-10. Solve the preceding problem if there is a constant inflow during this time of 400 c.f.s. *Ans.* 20.9 hrs.

28-11. If a navigation lock is 360 ft. long and 60 ft. wide, and the upper water surface is 16 ft. above the lower surface, what must be the total area of the ports if the lock is to empty in 5 min., if the loss of head through the ports and connecting culverts is such that the mean port velocity is one-third of the ideal velocity due to the difference in level? Since the area downstream from the lock will be very large, the lower level may be considered constant. *Ans.* 215 sq. ft.

CHAPTER IV

PIPE FLOW—I

29. "Pipe Friction."—The expression "pipe friction" is likely to give a wrong impression. The energy loss is not caused by the liquid rubbing along the inner surface of the pipe, like a swab being pushed through a gun barrel. The outer layer of molecules is practically at rest with respect to the pipe wall. But each layer inside, as we proceed toward the center of the pipe, is moving more rapidly, and the shearing deformation of the liquid caused by this relative motion does work on the liquid and converts

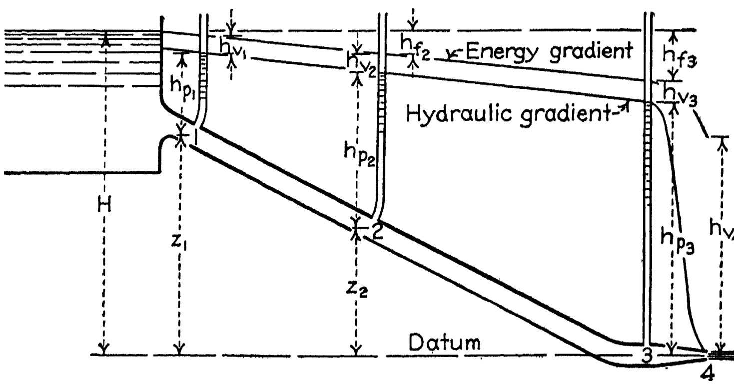


FIG. 63.—Energy and hydraulic gradients.

available energy into heat. In the case of turbulent flow there is also another more important cause of loss of energy, namely the continuous intermingling of portions of the liquid which are following irregular paths with differing velocities. The details of this matter are still imperfectly understood, but fortunately need not concern us here. (They are discussed to some extent in Appendix C.) All that needs emphasizing here is that the change from mechanical energy to heat takes place throughout the liquid, and not at the pipe wall. There is a force there but as it moves through no distance it does no work.

Figure 63 represents a pipe running from an elevated tank to a nozzle. Piezometers are attached at points 1, 2, and 3 to indicate the pressure. Measuring a distance h_v above this locates the energy gradient. It slopes downstream, the amount of the slope depending on the rate at which energy is being lost. The first quantitative study of this loss was by Couplet in 1732 (see Fig. 64), and the first formula to represent it seems to have

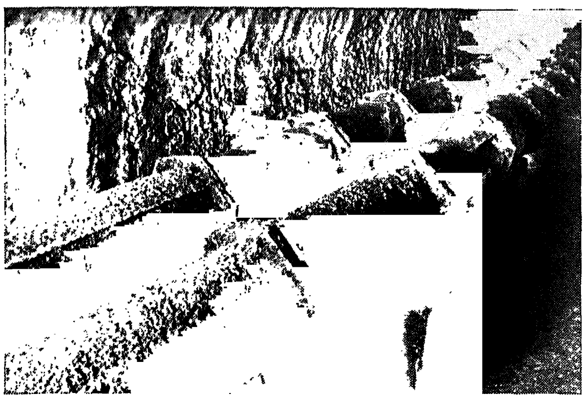


FIG. 64.—Cast-iron pipe installed in 1664 to supply water for fountains at Versailles, and experimented on in 1732 by Couplet. (Courtesy of Cast Iron Pipe Research Association.)

been proposed in 1775 by Antoine de Chezy (French, 1718–1798). This was in a form more adapted to flow in channels than to that in pipes; it is given in Chapter V. It was later restated by Darcy [1] in the form now generally used for pipes,

$$h_f = f\frac{L}{D}\frac{V^2}{2g} = f\frac{L}{D}h_v \tag{29.1}$$

in which L is the length of the pipe, D its diameter, and f is a dimensionless coefficient. For example, if $f = 0.02$ this means that 0.02 of a velocity head is lost for each diameter's length; if the pipe were 50 diameters long, the loss would be one velocity head; and if it were 5000 diameters long, h_f would be 100 velocity heads. This formula was also given by Weisbach, and is sometimes called the Weisbach formula. It is also often called the Fanning

[1] M. H. Darcy, *Recherches expérimentales au mouvement d'eau dans les tuyaux*, Paris, 1857.

formula, in honor of the author of the first textbook to popularize it in this country. The only reason for using the latter name is that both Darcy and Weisbach proposed other formulas also. Sometimes, however, the 2 in the denominator is omitted, or placed in the numerator, giving rise to three sets of values of f. In consulting other books the student should be on his guard as to which form is meant.

Experiments show that f is not constant, and that in general it depends on the diameter of the pipe, the mean velocity V, the density and viscosity of the liquid, and the roughness of the pipe walls. If the velocity is large enough, the properties of the liquid are found no longer to have any effect on f; the rougher the pipe walls, the lower the velocity necessary to make this true. A more complete treatment of the subject will be given in Chapter VI.

EXAMPLE I

If the pipe in Fig. 63 is 6 in. in diameter and 200 ft. long, and the jet from the nozzle is 2 in. in diameter and has a velocity of 90 ft. per sec., and if f is 0.020, find the loss of head from a point just within the entrance to the pipe to the beginning of the nozzle.

Since the diameter of the pipe is three times that of the nozzle, the velocity in the pipe will be $90 \div 9 = 10$ ft. per sec., therefore $h_v = \frac{100}{64.32} = 1.555$. $L = 200$ and $D = 6$ in. $= 0.5$ ft., therefore $h_f = \frac{0.02 \times 200}{0.5} \times 1.555$ $= 12.44$ ft.

EXAMPLE II

Taking the efficiency of the nozzle as 95 per cent, and assuming no loss at entrance to the pipe, find the total elevation of water surface above the nozzle in Example I (H in Fig. 63).

In the jet the total head is the velocity head $= \frac{90^2}{2\,g} = 8100 \div 64.32$ $= 125.9$ ft. This is 95 per cent of the total head at the beginning of the nozzle (point 3), so that there the total head must be $125.9 \div 0.95 = 132.5$. Therefore $H = 132.5 + 12.4 = 144.9$ ft.

EXAMPLE III

In Example II, find the pressure head at the beginning of the nozzle (point 3 in Fig. 63).

Velocity $= 10$ ft. per sec., therefore $h_v = \frac{100}{64.32} = 1.555$ ft. $h_p = 132.5$ $- 1.6 = 130.9$ ft.

PROBLEMS

29–1. Solve Examples I and II if the velocity of the jet were 72 ft. per sec.
Ans. 7.96 ft.; 92.8 ft.

29–2. Find the pressure head at the beginning of the nozzle in the preceding problem. *Ans.* 83.8 ft.

29–3. Make a scale drawing of the hydraulic and energy gradients for Prob. 29–1.

EXAMPLE IV

What would be the velocity of the jet, if the water surface in the above examples were 100 ft. above the nozzle?

Let V = velocity in pipe, then $9\,V$ = velocity of jet, and $\frac{V^2}{2\,g}$ and $\frac{81\,V^2}{2\,g}$ are the respective velocity heads. Then

$$100 = \frac{0.02 \times 200}{0.5} \times \frac{V^2}{2\,g} + \frac{81\,V^2}{2\,g \times 0.95} = (8 + 85.26)\,\frac{V^2}{64.32}$$

$V = \frac{80.2}{\sqrt{93.26}} = 8.30$ ft. per sec. in pipe. Therefore velocity of jet $= 8.30 \times 9 = 74.7$ ft. per sec.

PROBLEMS

29–4. Solve Example IV for $H = 121$ ft. *Ans.* 82.2 ft./sec.

29–5. What would be the velocity in the pipe in Example IV if the nozzle were removed and the pipe discharged directly into the open air? (H is still to be 100 ft., and assume that f is still 0.020.) *Ans.* 26.7 ft./sec.

29–6. What is the discharge in Example I? Check by finding it in both the pipe and the jet.

29–7. If the liquid is water, what is the total power in Example II? What per cent is lost in "pipe friction," and what per cent remains in the jet?
Ans. 32.4 hp.; 8.6%; 86.8%.

30. "f" in Rough Pipes.—Recently Nikuradse,[2] experimenting with pipes roughened by sand grains of definite size glued to the inside surface, found that for each pipe when V was more than a certain minimum value

$$\frac{1}{2\sqrt{f}} = \log_{10}\frac{r_0}{\epsilon} + 0.87 \qquad (30.1)$$

[2] J. Nikuradse, "Strömungsgesetze in rauhen Rohren," *Verein Deutscher Ingenieure*, Forschungsheft 361, 1933.

where r_0 is the radius of the pipe and ϵ (epsilon) is the mean diameter of the sand grains. Substituting D for $2\,r_0$, and solving for f, this gives

$$f = \frac{}{\left(1.138 + 2\log_{10}\frac{D}{\epsilon}\right)^2} \tag{30.2}$$

The ratio $\frac{D}{\epsilon}$ is usually called the *relative roughness*, although relative smoothness might be a more logical name, since it is larger the smoother the pipe. It is not to be supposed that the size of the particles is the only thing of importance, certainly sparsely spaced particles would not have the same effect as those closer together, and the effect of those with sharp edges would not be the same as those with smooth ones; but whatever the condition of the inner surface of any particular rough pipe, it has a value of f which is the same as the f for a pipe of the same diameter treated as Nikuradse's pipes with sand of some particular size. The diameter of these sand grains is called the *equivalent roughness* of the first pipe.[3] Figure 65 is a plotting of the values of f for various values of D and ϵ according to formula (30.2).

When one consults the records of tests on pipes actually used in engineering practice,[4] it is somewhat difficult to correlate them with Nikuradse's results. A much more thorough study of existing data and perhaps extensive new experiments will be needed before it is certain that a formula of the type of (30.2) will prove satisfactory for practical use. However, it has been shown by Prandtl and von Karman[5] to have a rational basis, and the following tentative values of ϵ (in feet) are given for use in working problems. They apply, it must be remembered, only to fairly high velocities. Just how high will be discussed in Chapter VI.

[3] See "Frictional Resistance in Artificially Roughened Pipes," by Dr. Victor L. Streeter, *Trans. A.S.C.E.* Vol. 101 (1936), pp. 681–713.

[4] Such as "Study of the Data on the Flow of Fluids in Pipes," by Emory Kemler, *Trans. A.S.M.E.*, HYD-55-2 (Aug., 1933), pp. 7–32.

[5] A summary (in English) of the work of these men is given in "Modern Conceptions of the Mechanics of Fluid Turbulence," by Dr. Hunter Rouse, *Trans. A.S.C.E.*, Vol. 102 (1937), pp. 463–543.

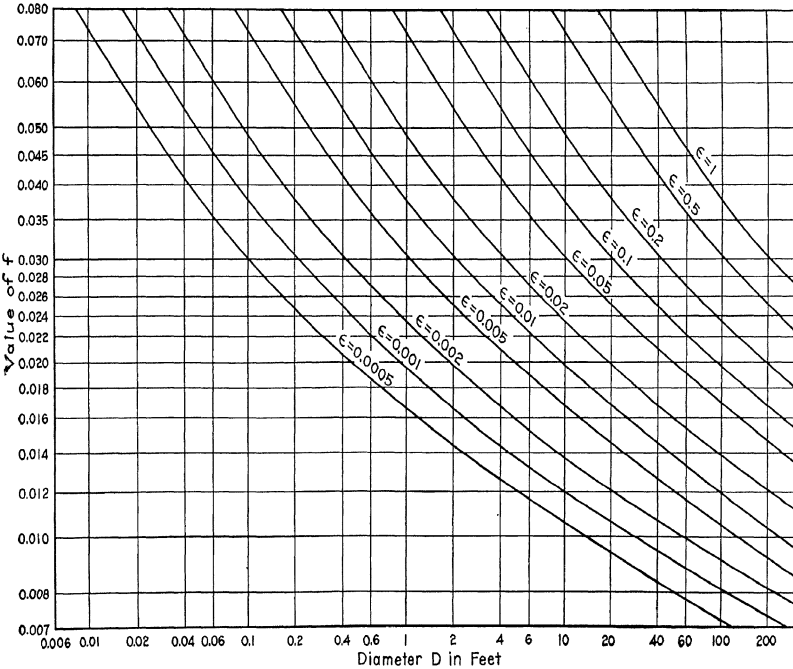


FIG. 65.—Graph of Nikuradse's formula for rough pipe.

TABLE I.—TENTATIVE VALUES OF ϵ (IN FEET)

MATERIAL	NEW	OLD
Wrought iron	0.0002	0.002
Galvanized iron	0.0005	0.02
Cast iron	0.0005	0.02
Good concrete tunnel lining	0.001	0.002
Drain tile	0.002	0.005
Riveted steel	0.002	0.02
Concrete pipe (jointed)	0.005	0.01

The expression "old" is, of course, very indefinite. Rules have been proposed for giving f at the end of any given number of years. But deterioration of metal pipe is not a simple function of the time. It depends very much on the liquid flowing. In the case of water it depends on the amount of carbon dioxide present,[6]

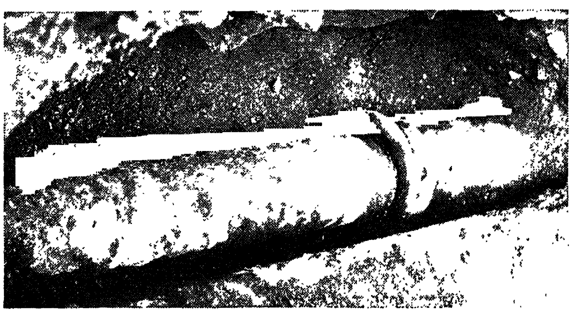


FIG. 66.—Twelve-inch cast-iron water main installed in Lynchburg, Va., in 1829 and still in service. (Courtesy of Cast Iron Pipe Research Association.)

and on the presence or absence of electric currents which might cause electrolysis. In the above table "old" may be taken to mean pipe in need of cleaning, or near the end of its useful life. Since there is so much uncertainty as to the correct value of f, it should be taken to only two significant figures in the problems. It can therefore be read from Fig. 65 without use of formula (30.2).

A vast number of empirical formulas have been put forward for computing the loss of head in pipes, some of them quite satis-

[6] For a case in which it proved advisable to spend thousands of dollars to remove the carbon dioxide, see *Eng. News-Record,* Vol. 120 (June 23, 1938), pp. 885–886.

factory for the range of values for which they were derived. They have the advantage of applying to lower velocities than equations (30.1) and (30.2), but they are all untrustworthy for high velocities and large sizes, as in Probs. 30–4 and 30–5 below. Most of these formulas may be put in the form

$$(30.3)\quad h_f = k_1 L \frac{V^n}{D^m} = k_2 L \frac{Q^n}{D^{m+2n}}$$

Values of n vary from 1.72 for very smooth pipe to 2.00 for very rough pipe; m is generally taken at from 1.16 to 1.25; and the values of k_1 and k_2 vary widely with the kind of pipe. One of the most popular of these formulas, known as the Hazen-Williams[7] formula from the names of its authors, is

FIG. 67.—Concrete pipe. (Courtesy Lock Joint Pipe Co.)

$$(30.4)\qquad V = 1.318\, C\, R^{0.63} S^{0.54}$$

where C is an empirical constant, R is the hydraulic radius, and S is the slope of the energy gradient. As explained in Art. 41, R is here $D/4$, so that the formula may also be written,

$$(30.5)\qquad V = 0.550\, C\, D^{0.63} S^{0.54}$$

C is given as 130 for new cast-iron pipe, 120 for cast-iron pipe after from 4 to 6 years of use,[8] and 100 for cast-iron pipe after from 15 to 20 years of use, also for large steel pipe 10 years old, and for brick sewers. A chart for solving this formula will be found in Appendix A.

[7] For extensive tables for the use of this formula, see *Hydraulic Tables* by G. S. Williams and Allen Hazen, John Wiley & Sons (1914). Special slide rules for solving this formula have also been put on the market.

[8] Actually C varies somewhat with the size of the pipe as well as with the years of service. As pointed out in Chapter VI, it is relative rather than absolute roughness that governs.

A much used formula[9] for the flow in wood-stave pipe takes the form of equation (30.3) with $n = 1.80$, $m = 1.17$, and $k_1 = 0.000419$. These formulas were proposed before the action of wall roughness as discussed in Chapter VI was understood. The writer believes that they will gradually fall into disuse as more rational methods are developed.

EXAMPLE I

Find the value of f to use in Example II of the previous article, if the pipe is (a) old cast iron, (b) new cast iron.

From the table, ϵ for old cast iron $= 0.02$ ft., $\dfrac{D}{\epsilon} = \dfrac{0.5}{0.02} = 25$, $\log_{10} 25$ 1.398 and $f = \dfrac{1}{(1.138 + 2.796)^2} = \dfrac{1}{15.48} = 0.065$. This checks on Fig. 65.

ϵ for new cast iron $= 0.0005$, $\dfrac{D}{\epsilon} = 1000$, $\log_{10} 1000 = 3.000$ and $f = \dfrac{1}{7.138^2} = 0.0196$, so that the assumption of 0.020 happened to be very close.

PROBLEMS

30–1. Check the derivation of equation (30.2) from (30.1).

30–2. Solve Prob. 29–5 if the pipe is old galvanized iron. *Ans.* 15.4 ft./sec.

30–3. Estimate the loss of head per 1000 ft. of a new 9 ft. riveted steel pipe when the velocity is 15.45 ft. per sec. *Ans.* 5.77 ft.

Note. This is Pipe No. 230 of Scobey's tests. Measurements gave 5.761 ft.

30–4. The diversion tunnels at Boulder dam were concrete lined and 50 ft. inside diameter. Estimate the loss of head per 1000 ft. for a velocity of 50 ft. per sec. *Ans.* 7.00 ft.

30–5. The main penstocks at Boulder dam are of riveted steel and 30 ft. inside diameter. Estimate the loss of head per 1000 ft., when old, for a flow of 25,000 c.f.s. *Ans.* 11.7 ft.

Minor Losses

31. Loss at Enlargement.—Besides the loss of head just considered, which is uniformly distributed along the pipe, there are sometimes losses which are concentrated at certain points. These are called *minor losses*, because in a long pipe line they represent

[9] *The Flow of Water in Wood-stave Pipe*, by Fred C. Scobey, U. S. Department of Agriculture, Bull. No. 376 (1916).

a small per cent of the whole loss. (However, in a short pipe they may comprise a large per cent of the total loss.) One such loss, is that at a sudden enlargement.

In Fig. 68, the cross-sectional area of the pipe suddenly increases from A_1 to A_2 and the pressure head increases from h_{p1} to h_{p2}. But this increase in pressure is not as great as Bernoulli's theorem without loss would give, as there is a definite loss of energy at the expansion. The magnitude of this loss can be deduced from the principle of the conservation of momentum. The momentum passing section 1 per second is $\frac{w\,A_1V_1^2}{g}$, and that passing section 2 is $\frac{w\,A_2V_2^2}{g}$. Therefore the loss in momentum per second is $\frac{w}{g}\,(A_1V_1^2 - A_2V_2^2)$. This must equal the unbalanced force on the mass of liquid *abcdefgh*. If the unit pressure on the annular area *bcfg* is the same as the pressure at 1, and experiment shows it is at least very nearly the same, this unbalanced force is $p_2A_2 - p_1A_2$. Then

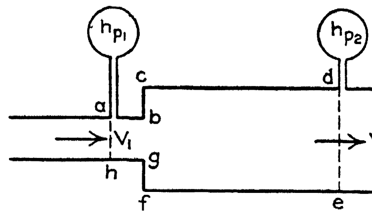


FIG. 68.—Sudden expansion.

$$(p_2 - p_1)\,A_2 = \frac{w}{g}\,(A_1V_1^2 - A_2V_2^2)$$

But by Bernoulli's theorem $\frac{p_1}{w} + \frac{V_1^2}{2g} = \frac{p_2}{w} + \frac{V_2^2}{2g} + h_f$, therefore

$$h_f = \frac{V_1^2}{2g} - \frac{V_2^2}{2g} + \frac{p_1 - p_2}{w} = \frac{V_1^2}{2g} - \frac{V_2^2}{2g} - \frac{(A_1V_1^2 - A_2V_2^2)}{g\,A_2}$$

which reduces to $\quad h_f = \frac{V_1^2}{2g} - \frac{A_1V_1^2}{A_2g} + \frac{V_2^2}{2g}$

But from the law of continuity

$A_1V_1 = A_2V_2,\quad$ therefore $\quad \frac{A_1V_1^2}{A_2g} = \frac{A_2V_2V_1}{A_2g} = \frac{2\,V_1V_2}{2g},\quad$ and

$$h_f = \frac{(V_1 - V_2)^2}{2\,g} \tag{31.1}$$

That is, the loss of head at a sudden enlargement is the head corresponding to the difference between the two velocities. This law, deduced from the principle of conservation of momentum, is verified very closely by experiment. (The student must be very clear as to the distinction between $(V_1 - V_2)^2$ and $V_1^2 - V_2^2$.)

EXAMPLE

If the velocity in the small pipe is 18 ft. per sec., and the large pipe has three times the diameter of the small pipe, find the head lost in the sudden enlargement.

The large pipe has nine times the area of the small pipe, therefore the velocity in the large pipe is 2 ft. per sec. Then $18 - 2 = 16$, and $h_f = \frac{16^2}{64.32} = 3.98$ ft.

For gradual expansions, such as the downstream part of a Venturi meter or in the draft tube of a turbine, we may write

$$h_f = \frac{k' (V_1 - V_2)^2}{2g} \tag{31.2}$$

where k' is, of course, generally less than unity. Strange to say, however, for cones in which the elements make 30° with the axis, Gibson [10] has found k' to be about 1.16. When the elements make an angle of 3° with the axis, k' is about 0.13.

PROBLEMS

31–1. A 6-in. pipe suddenly expands into a 12-in. pipe. If the velocity in the 6-in. pipe is 40 ft. per sec., what is the loss of head at the expansion? *Ans.* 14.0 ft.

31–2. What would the loss have been if the expansion had been gradual with $k' = 0.20$? *Ans.* 2.80 ft.

31–3. If the pressure in the 6-in. pipe of Prob. 31–1 was 40 ft., what was the pressure in the 12-in. pipe? *Ans.* 49.3 ft.

31–4. Make a scale drawing of the energy and hydraulic gradients in the preceding problem, assuming that the velocity becomes distributed across the cross-section at a distance of about three diameters downstream from the enlargement, and that f is 0.03 in both pipes.

[10] *Trans. Royal Soc. A*, Vol. 83 (1910), pp. 366–378; or *Engineering*, Vol. 93 (Feb. 16, 1912), pp. 205–206.

32. Loss at Entrance.—Figure 69 shows a pipe with a flush entrance. The square corner acts as a sharp-edged orifice and forms a jet at point 1 with an area of about 60 per cent of the area of the pipe. The space outside the jet is filled with liquid which takes no part in the flow; that is, it is practically a stationary eddy. There is very little loss up to point 1, but from there to 2 there is loss by expansion, as in the preceding article.

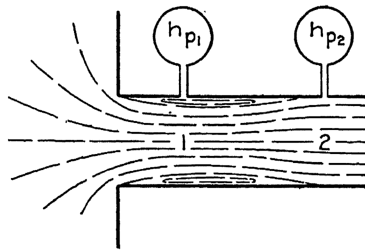


FIG. 69.—Abrupt entrance.

If V_1 is the velocity in the jet and V_2 is the velocity in the pipe at 2 and beyond,

$$V_1 = \frac{V_2}{0.60} = 1.667\, V_2, \quad \text{and} \quad V_1 - V_2 = 0.667\, V_2, \quad \text{therefore,}$$

assuming that the loss is the same as at a sudden expansion, we have

$$h_f = \frac{(V_1 - V_2)^2}{2g} = 0.44 \frac{V_2^2}{2g} \tag{32.1}$$

Actual experiments on the loss at this sort of entrance give values generally a little larger than this, and since entrance loss is usually a very small part of the total, it is usual to use the general formula

$$h_f = k \frac{V_2^2}{2g} \tag{32.2}$$

with the round value of 0.50 for k for a square-edged opening.

As shown in Art. 21, when the pipe is thin and extends back into the tank one diameter or more, C_c is 0.50. From this by the same reasoning as above, the student may easily prove that k would be 1.00. But, of course, all actual pipes have some thickness, so that the loss will never be as great as one velocity head. In fact, as stated in Art. 21, all standard pipes (if cut square) 5 inches or less in diameter act as flush entrances. The values

of k for re-entrant entrance into other standard steel or wrought-iron pipes up to 12 inches, computed on the assumptions made above, are given in the following table. Actual values are probably a little larger.

Diameter			C_c	k
Nominal	Actual Inside	Actual Outside		
6	6.065	6.625	0.597	0.46
8	8.071	8.625	0.571	0.56
10	10.192	10.750	0.556	0.64
12	12.000	12.750	0.565	0.59

Larger steel pipes are generally relatively thinner and k is larger up to, say, 0.90. Often for simplicity and to be on the safe side, k for a re-entrant entrance is taken as 1.00.

Rounded or bell-mouthed entrances as shown in Figs. 52 and 63, give a jet the full size of the pipe. Therefore, the only loss is the drag on the curved surface. It may be taken as 0.04 of the velocity head in the pipe, or in most cases it may be neglected entirely.

EXAMPLE I

In Example II of Art. 29, how much greater would H have been if the entrance had been flush?

The velocity head in the pipe as given in Example I of that article was 1.555. Then taking k as 0.50, the head lost at entrance would have been 0.78 ft. This would have increased H from 144.9 ft. to 145.7 ft.

EXAMPLE II

Check the computation of the first line of the above table.

The area of the circle whose diameter equals the outside diameter of the pipe is $\dfrac{\pi \times 6.625^2}{4} = 34.47$ sq. in. 0.50 of this is 17.24 sq. in. This is less than 0.60 of the inside area, so the outside corner governs. Therefore

$$C_c = \frac{17.24}{28.89} = 0.597. \qquad h_f = \frac{(V_1 - V_2)^2}{2g} = \frac{\left(\frac{V_2}{C_c} - V_2\right)^2}{2g} = \left(\frac{1}{C_c} - 1\right)^2 \frac{V_2^2}{2g}.$$

Therefore $k = \left(\dfrac{1}{C_c} - 1\right)^2 = (1.675 - 1)^2 = 0.4556$. This was taken as 0.46.

PROBLEMS

32-1. Solve Prob. 29-5 if the pipe had a flush entrance.

Ans. 26.0 ft./sec.

32-2. Show that $k = 1$ for a re-entrant entrance to a very thin pipe.

32-3. Check the value of k for the 12-in. pipe in the above table.

32-4. The so-called "standard short tube" consists of a flush entrance and a pipe three diameters long from which the jet issues the full size of the tube. Taking k as 0.444 and f as 0.020, find the coefficient of discharge and the efficiency. *Ans.* 0.815; 0.665.

32-5. Find the value of k for a re-entrant entrance to a standard 16-in. pipe (16-in. outside diameter and 3/8-in. thickness). *Ans.* 0.67.

32-6. A 57-in. piece of 15-in. pipe (inside diameter = 14.25, outside diameter = 15.00) extends back half its length into a large tank. The surface of the water is 36 ft. above the center of the pipe. Calculate the flow (in c.f.s.) if the jet comes through without touching the downstream end of the pipe. Also calculate the flow if the downstream end of the pipe flows full and $f = 0.025$. *Ans.* 29.5 c.f.s.; 40.3 c.f.s.

EXAMPLE III

A 6-in. piece of standard 1-in. pipe (inside diameter = 1.049, outside diameter = 1.315) is cut off square and inserted into the side of a large tank into which water at 100° F. is flowing faster than it flows out of this pipe. At first the pipe flows full, but at a certain head on the pipe the jet springs clear through the pipe without touching the sides. Estimate what this head was, and what the flow was just before and just after the change.

When the pressure at the contraction becomes so small that vapor begins to form, it will blast an opening out to the air, and air will flow back around the jet. Steam tables give the pressure of water vapor at 100° F. as 0.946 lb., per sq. in., or 136.2 lb. per sq. ft. Water at 100° F. weighs 62.0 lb. per cu. ft. therefore the head is 2.20 ft. of water. One atmosphere = 2116 lb. per sq. ft. and 2116 ÷ 62 = 34.13 ft. of water of this weight. Assuming that the area of the jet is 0.60 the area of the pipe, calling the head on the pipe H, and writing Bernoulli's equation from the free surface to the contracted jet, pressures being above absolute zero, we have

$$H + 34.13 = 2.20 + \frac{V_1^2}{2g} = 2.20 + \frac{(1.667\,V_2)^2}{2g}$$

When the pipe is running full

$$H = \frac{V_2^2}{2g} + \frac{0.5\,V_2^2}{2g} + \frac{0.12\,V_2^2}{2g} = \frac{1.62\,V_2^2}{2g}$$

(The $\frac{0.12\,V_2^2}{2g}$ is added as an estimate of the "pipe friction" in 6 diameters

of pipe, with $f = 0.020$.) Subtracting the second equation from the first we have $34.13 = 2.20 + (2.78 - 1.62)\frac{V_2^2}{2g}$ or $\frac{V_2^2}{2g} = \frac{31.93}{1.16} = 27.52$. Therefore, $H = 1.62 \times 27.52 = 44.58$ ft. The inside area of the 1-in. pipe is $\frac{\pi \times 1.049^2}{4} = 0.8626$ sq. in. $= 0.00599$ sq. ft. $V_2 = 8.02\sqrt{27.52} = 42.1$ ft. per sec. While the pipe is still running full, $Q = 0.00599 \times 42.1 = 0.252$ c.f.s. When the jet breaks clear, $Q = 0.60 \times 0.00599 \times 8.02\sqrt{44.58} = 0.02882 \times 6.677 = 0.192$ c.f.s.

PROBLEM

32-7. Solve Example III for gasoline weighing 44 lb. per cu. ft. and a vapor pressure of 5.0 lb. per sq. in.

Ans. 44.3 ft.; 0.251 c.f.s.; 0.192 c.f.s.

33. Loss at Contraction.—When a large pipe suddenly contracts into a smaller one, as in Fig. 70, a jet forms and then expands again, very much as at a flush entrance. In fact the flush entrance is simply the limiting case of sudden contraction, as the first pipe gets larger and larger. In the case shown in Fig. 70, the restricted approach interferes with the contraction of the jet and C_c is more than 0.60. Therefore, the loss is less than for an abrupt entrance. Experiment shows that the loss at a sudden contraction is very well represented by $k\frac{V_2^2}{2g}$ where

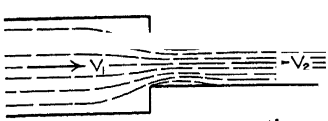


FIG. 70.—Sudden contraction.

$$k = \frac{4}{9}\left(1 - \frac{A_2}{A_1}\right) \tag{33.1}$$

A_2 being the cross-sectional area of the small pipe, and A_1 that of the large one. When A_1 is very large compared with A_2, this gives $k = 0.44$ as in equation (32.1).

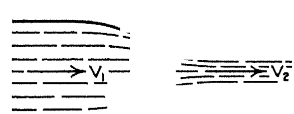


FIG. 71.—Gradual contraction.

If the reduction of area is made gradually, either by introducing a cone or a curved reduction, as in Fig. 71, the loss of head becomes very small (k = about 0.04 in the latter case).

EXAMPLE

Twenty ft. of 12-in. pipe leads out of a tank by a flush entrance, and then contracts suddenly to a 6-in. pipe. There is 50 ft. of the latter and it discharges directly into the atmosphere. How high must the free surface in the tank be above the discharge end of the pipe in order that the velocity in the 6-in. pipe be 16 ft. per sec.? (Take f as 0.017 in both pipes.)

The velocity in the 12-in. pipe will be $16 \div 4 = 4$ ft. per sec. Therefore, the velocity head in the 12-in. pipe is $\frac{4^2}{64.32} = 0.249$ ft. Then the head lost at entrance will be 0.124 ft. The loss in the 12-in. pipe will be $\frac{0.017 \times 20}{1} \times 0.249 = 0.085$ ft. By (33.1), k for the contraction will be $\frac{4}{9}(1 - \frac{1}{4}) = \frac{1}{3}$. The velocity head in the 6-in. pipe is $\frac{16^2}{64.32} = 3.98$ ft. Therefore, the loss at contraction is $\frac{3.98}{3} = 1.327$ ft. The loss in the 6-in. pipe is $\frac{0.017 \times 50}{0.5} \times 3.98 = 6.77$ ft. Then the total head required is

0.12	+ 0.08	+ 1.33	+ 6.77	+ 3.98	= 12.28 ft., or, say, 12.3 ft.
entrance	1st pipe	contraction	2nd pipe	h_v at exit	

The major uncertainty is the friction loss in the smaller pipe. As this would probably not be known within nearer than 6 per cent of its true value, the error in the 6.77 term may be as much as 0.4 ft., so that there is little use in carrying the answer beyond tenths of a foot.

PROBLEMS

33-1. What would k be for a sudden contraction from a 12-in. pipe to an 8-in. pipe? *Ans.* 0.25

33-2. What head would be lost at the contraction in the preceding problem, if the velocity in the 12-in. pipe were 4 ft. per sec.? *Ans.* 0.31 ft.

33-3. Solve the two preceding problems if the 8-in. pipe is changed to a 4-in. pipe. *Ans.* 0.395; 7.96 ft.

33-4. What head would be lost if the pipe in the example were reversed; that is, let the 6-in. pipe take off from the tank and enlarge into a 12-in. pipe? Why is the answer less than in the example? *Ans.* 11.3 ft.

34. Other Minor Losses.—When a pipe discharges into a tank the velocity head of the stream is soon dissipated by the disturbed flow in the tank. This is true whether the exit is submerged or not. So the 3.98 ft. term in the preceding example is often called the *loss at exit*. This is allowable if the Bernoulli equation

is being written to the tail-water and all of the original head is accounted for by losses. But if Bernoulli's theorem is written to a point in the jet which still has the velocity V_2, the $\frac{V_2^2}{2g}$ term remains as energy and there has been no exit loss. The equation is exactly the same in either way of looking at the matter. The only danger is that the student may put the term in twice, once as a loss, and then as energy still available.

Another minor loss is that due to curvature or bends in the pipe. This loss may be expressed in two ways: (1) The loss in head caused by a bend, over and above the loss due to ordinary pipe friction in the same length of straight pipe, may be written as $k_c \frac{V^2}{2g}$, where k_c depends upon several factors including the total angle of the bend, the ratio of the radius of the bend to the pipe diameter, and the internal roughness. (2) The effect of curvature may be stated as a "curvature length," defined as the length of straight pipe in which the friction loss would be the same as the extra loss in the bend; that is, for which $L = \frac{D\,k_c}{f}$. Since k_c varies with the roughness of the pipe and the velocity, in somewhat the same way as does f, this second method has the advantage that one table or graph may be used for approximate results for all cases. This latter is the method usually used in practice. Actually, however, the curvature length depends also upon other factors. There are many experimental data on the loss in bends, but the agreement between different experimenters is not satisfactory. Figure 72 is based on results given by Albert Hofmann,[11] who seems to have gone into the matter very carefully. The upper curve is for 90° elbows having the inner surface painted with a sand and paint mixture which gave an f for straight pipe of about 0.024, and, therefore, by equation (30.1) would have an $\frac{r_0}{\epsilon}$ of about 220. Most actual bends in pipes would be smoother than this and the loss would be somewhat less. The lower curve is for a 90° elbow made very smooth on the inside,

[11] "Loss in 90-degree Pipe Bends of Constant Circular Cross-section" in *Trans. of the Hydraulic Institute of the Munich Technical University*, Bull. No. 3, pp. 29–41, published (in English) by the A.S.M.E. (1935).

and is for a Reynolds number of about 200,000 (the meaning of this is explained in Art. 50). The equivalent length for a Reynolds number of 100,000 was found to be about 12 per cent greater.

In the extreme case of a mitre bend and probably also in bends of unusually small radius the liquid on the inside of the bend is unable to "turn the corner" suddenly, and a back eddy forms which reduces the effective area in much the same way as occurs at an abrupt entrance. As in that case, the energy loss is due chiefly to the expansion downstream from the contraction. For bends of larger radius no contraction occurs, but the liquid on the inside of the curve moves faster and that on the outside slower. In fact, the liquid in the bend forms very nearly a free vortex, with the velocities varying inversely with the radius of curvature of the path. When the liquid has passed the bend, it does not immediately return to its condition of normal velocity distribution. Until it does so, the rate of loss of head is greater than normal. Therefore, the loss due to the bend does not all occur within the bend, but extends downstream from 20 to 60 pipe diameters.[12]

Due to this fact, two 90° bends joined to form a 180° bend do not give as great a loss as though separated by, say, 60 diameters of straight pipe. David L. Yarnell [13] found that a 180° bend gave 1.50 times the loss of a 90° bend, and a 45° bend 0.75 times that for a 90° bend.

It may also be noted that due to the effect discussed in Art. 19, the pressure on the outside of a bend is greater than that on the inside. Experiments have shown the correlation between this pressure difference and the average velocity in the cross-section, so that an estimate of the flow may be made from the observed difference in pressure between taps on the outside and inside of the bend.[14]

[12] For a thorough discussion of this, see Technical Bull. No. 577, U. S. Dept. of Agriculture.

[13] "Flow of Water through Pipe Bends," *Eng. News-Record*, June 2, 1938, pp. 777–778.

[14] Herbert Addison, "The Use of Pipe Bends as Flow Meters," *Engineering*, Mar., 1938, pp. 227–229. Also Wallace M. Lansford, *The Use of an Elbow in a Pipe Line for Determining the Rate of Flow in the Pipe*, Bull. No. 289, Univ. of Illinois, Eng. Exp. Station.

It should be emphasized that Fig. 72 represents the *extra* loss due to curvature. The "total equivalent length" of a bend is the sum of its actual length (measured along the center line) and its

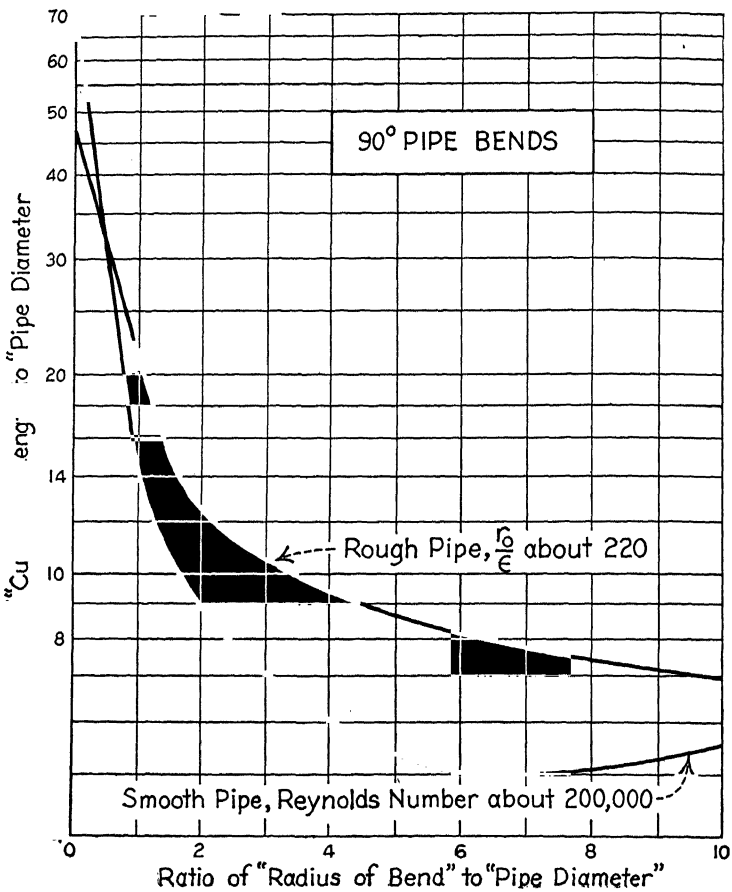


FIG. 72.—Loss in 90° pipe bends.

"curvature length." On the other hand it must be remembered that a 90° arc is shorter than two tangents plus a short bend, and a 90° arc with a radius of 10 diameters shows less total loss than do two tangents 9 diameters long and an arc of one diameter.

EXAMPLE

Using the curve for rough pipe in Fig. 72, check the statement made just above.

The problem is illustrated by Fig. 73. The distance around the long arc is 15.7 D. Therefore the total equivalent length is (15.7 + 6.9) $D = 22.6\ D$. The length of arc BC is $1.6\ D$, $AB = 9\ D$, and $CE = 9\ D$, therefore the length of that path is $19.6\ D$. But by Fig. 72, the "curvature length" is $21\ D$, therefore the total equivalent length is $40.6\ D$, which is considerably more than for the long arc. (The cost of the long arc might also be greater.)

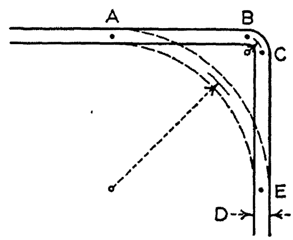


FIG. 73.

PROBLEMS

34–1. Make a similar check comparing 10 D with 6 D.
Ans. 6 D has 13% greater total equivalent length than 10 D.

34–2. Find the total loss in head in a 90° bend in a rough 2-in. pipe, for which $\frac{r_0}{\epsilon} = 220$, if the radius of the bend is 6 in., and the mean velocity 8 ft. per sec. *Ans.* 0.365 ft.

34–3. What was k_c in the preceding problem? *Ans.* 0.254.

34–4. The value of $k_c = 0.75$ is often given for a threaded elbow. If the radius of the bend equals the diameter of the pipe, and the rough pipe curve of Fig. 72 applies, what is the value of f? *Ans.* 0.035.

NOTE. This answer would seem to indicate that the figure 0.75 is too high. But it must be remembered that a threaded elbow comprises not only a bend, but also an expansion and a contraction.

34–5. Estimate the loss due to curvature in a 45° bend in an old concrete-lined tunnel 10 ft. in diameter, if the radius of the bend is 50 ft., and the mean velocity is 80 ft. per sec. *Ans.* 9.1 ft.

34–6. Construct a curve similar to the rough pipe curve of Fig. 72, but giving the ratio of "total equivalent length" to "pipe diameter" for 90° pipe bends. For what radius is this a minimum? Is this the best radius of bend to use?

35. Pipe Lines.—Under this heading we shall consider problems involving various combinations of the items considered in the last few articles.

If the lengths, diameters, and roughness of the various parts of the line are given, as well as the flow (or what amounts to the same thing, the velocity at some point of the pipe), and the problem is to find the total head required, the solution is straight-

forward. All that is necessary is to add up the losses, as discussed above. However, where different parts of the line are of different diameters, the velocity in each part must first be found (by the law of continuity) and care must be used that each loss is based on the proper velocity. See Example I.

If the flow is unknown but the total head and all the other items listed above are given, let h_v be the velocity head at any one section (usually in the largest pipe) and express the other velocity heads in terms of it, remembering that they vary inversely as the fourth power of the diameter. Write out the equation for total head, solve for h_v, and from it get V and Q. See Example II. If the pipe is smooth enough, or the velocity low enough, so that the values of Art. 30 do not apply, the methods of Chapter VI must be used and it will be necessary to work by successive approximations, making a first guess as to the velocities to get a first trial for f, and then revising it if necessary. Such problems will be avoided in this chapter or an arbitrary value of f will be given without specifying the kind of pipe.

When the total head, the length, and the flow are given, and the problem is to find the necessary size of pipe, the method is somewhat different. In this sort of problem, which often confronts the engineer in practice, the minor losses are usually such a small part of the whole that they may be neglected, and, since $V = \frac{4\,Q}{\pi\,D^2}$,

$$(35.1) \qquad H = \frac{f\,L}{D \times 2\,g} \times \frac{16\,Q^2}{\pi^2\,D^4} = \frac{f\,L\,Q^2}{39.7\,D^5}, \qquad \text{or}$$

$$(35.2) \qquad D^5 \quad \frac{f\,L\,Q^2}{39.7\,H}$$

A first guess will have to be made as to f, which will give a first approximation of D. From it a more accurate value of f can be obtained which will give a sufficiently accurate value of D. Ordinarily the engineer is limited to commercial sizes of pipes. If, as is usually the case, the head given is a maximum and the flow a minimum and the computed value of D falls between two sizes, the next larger is taken, so that the required flow will be delivered by a head slightly less than the available head; or, using the whole head will give slightly more than the required flow. Under some

circumstances, however, the commercial size nearest the computed D should be used. See Example III.

These are the only cases that will be considered in this article. Others are taken up in the following three articles.

EXAMPLE I

A 2-in. pipe takes off abruptly from a large tank, goes 20 ft., and then expands abruptly to 4-in. diameter, runs for 100 ft., and discharges directly into the open air with a velocity of 4 ft. per sec. Compute the necessary height of water surface above the point of discharge if f for the pipes is 0.026.

Velocity in 2-in. pipe $= 4 \times 4 = 16$ ft. per sec., therefore, velocity head $\frac{16^2}{64.32}$ 3.98 ft. and velocity head in 4-in. pipe $= \frac{4^2}{64.32} = 0.249$ ft.

$$
\begin{array}{lr}
\text{Loss at entrance} = 0.50 \times 3.98 & = 1.99 \text{ ft.} \\
\text{Loss in 2-in. pipe} = 0.026 \times 20 \times 6 \times 3.98 & = 12.42 \text{ ft.} \\
\text{Loss at expansion} = \dfrac{(16-4)^2}{2g} = \dfrac{144}{64.32} & = 2.24 \text{ ft.} \\
\text{Loss in 4-in. pipe} = 0.026 \times 100 \times 3 \times 0.249 & = 1.94 \text{ ft.} \\
\text{Loss at exit} & = 0.25 \text{ ft.} \\
\text{Necessary height} & 18.84 \text{ ft.}
\end{array}
$$

EXAMPLE II

Find the velocity of flow in the 4-in. pipe of Fig. 74.

Let h_v = velocity head in 4-in. pipe. Then $16\,h_v$ = velocity head in 2-in. pipe. By (33.1), k for the contraction is 0.333. The 4-in. pipe is so thick that the entrance is flush. Then

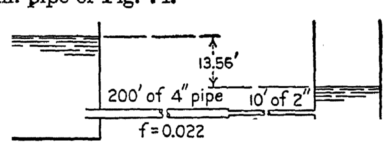


FIG. 74.

$$13.56 = (0.5 + 0.022 \times 200 \times 3)\,h_v + (0.333 + 0.02 \times 10 \times 6 + 1)\,16\,h_v$$
$$13.56 = 13.7\,h_v + 2.533 \times 16\,h_v = 54.23\,h_v.$$

Therefore,

$$h_v = \frac{13.56}{54.23} = 0.250, \qquad \text{and} \qquad V = 8.02\sqrt{0.25} = 4.01 \text{ ft. per sec.}$$

EXAMPLE III

Find the necessary diameter of old riveted steel pipe, to carry 4000 gallons of water per minute with a total loss of 40 ft. of head per 1000 ft. of pipe.

One c.f.s. equals 448.8 U. S. gallons per min., therefore $Q = 4000$

$\div$ 448.8 = 8.91 c.f.s. Then $Q^2 = 79.4$. Assume $f = 0.025$. Then by (35.2), $D^5 = \dfrac{0.025 \times 1000 \times 79.4}{39.7 \times 40} = 1.250$. By log-log slide rule, $D = 1.046$ ft. (It can also be obtained by logarithms.) Therefore a 12-in. pipe would not be large enough, especially since Fig. 65 shows that our assumed value of f was too small. The next usual size of pipe would be 14-in., for which $D = 1.167$ ft. For this, and $\epsilon = 0.020$, Fig. 65 gives $f = 0.046$. Then

$$H = \frac{0.046 \times 1000 \times 79.6}{39.7 \times 2.16} = 42.8 \text{ ft.}$$

Therefore, even this is not quite sufficient and we will try 16-in. For this $f = 0.044$ and $H = \dfrac{0.044 \times 1000 \times 79.4}{39.7 \times 4.21} = 21.0$ ft. Therefore, this is much larger than necessary. Our value $\epsilon = 0.020$ is, of course, very arbitrary and the 14-in. pipe would doubtless meet the requirements for many years after its installation.

PROBLEMS

35–1. Solve Example I for old wrought iron pipe. *Ans.* 26.4 ft.

35–2. Water enters a 6-in. pipe from a large tank by a flush entrance, and flows for 100 ft. Then the pipe enlarges abruptly to a 12-in. pipe and flows for 200 ft., and finally discharges freely into the air. Estimate the discharge if the water in the tank is 36 ft. above the discharge end of the pipe. Take $f = 0.020$ in both parts of the pipe. *Ans.* 4.07 c.f.s.

35–3. Gasoline flows from one tank into a 2-in. pipe, 10 ft. long, then expands abruptly into a 4-in. pipe, which is 50 ft. long, and then discharges into another tank, the exit being submerged. If the free surface is 10 ft. lower in the second tank than in the first, and f is 0.020 in both pipes, find the velocity in the 4-in. pipe. *Ans.* 4.00 ft./sec.

35–4. Solve the preceding problem if the pipes are reversed, the flow being first through the 4-in. pipe, and then by abrupt contraction to the 2-in. pipe. *Ans.* 3.82 ft./sec.

35–5. One hundred ft. of standard 4-in. pipe (see Table XIII in Appendix A) runs from an elevated tank to a point 40 ft. below the free surface, where it discharges directly into the open air. It contains two elbows (each with $k_c = 0.75$) and projects into the tank. If $f = 0.021$, what is the discharge? *Ans.* 1.47 c.f.s.

35–6. Water flows from an elevated tank through 40 ft. of 6-in. pipe, and then through 120 ft. of 12-in. pipe. Then it discharges into the air through a nozzle which gives a 4-in. jet. The entrance is well rounded, the enlarge-

ment from 6 in. to 12 in. is abrupt, the loss in the nozzle may be taken as 6 per cent of the velocity head of the jet, and f for both pipes may be taken as 0.025. The water surface in the tank is 100 ft. above the nozzle. Find the velocity in the 12-in. pipe. *Ans.* 7.04 ft./sec.

35–7. What is the efficiency of the above nozzle? What is the pressure head at the base of the nozzle? *Ans.* 94.3%; 65.3 ft.

35–8. Make a scale drawing of the energy and hydraulic gradients for Prob. 35–6.

35–9. Figure 75 represents a pump P, pumping a liquid from a tank at a velocity of 8 ft. per sec. through a 3-in. pipe. Point A is 5 ft. above the free surface, B is 7 ft. above the free surface, the length of pipe from entrance to A is 10 ft., the total length from B to the exit is 40 ft., the radius of the bends is 6 in., and f is 0.024. Find the pressure head at A and at B and the head against which the pump operates. *Ans.* – 7.45 ft.; 22.42 ft.; 31.87 ft.

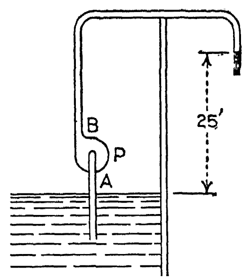


FIG. 75.

35–10. Solve Example III for new riveted steel pipe.

35–11. If a 14-in. pipe is used in Example III, estimate the flow in gal. per min. for a 40 ft. head, when the pipe is new, and when it is old. *Ans.* 5480 gal./min.; 3880 gal./min.

36. Pipe Networks.—When two or more pipes are acting in parallel (that is, they have the same source and the same terminus) and the difference in pressure between the source and terminus is known, the flow in each pipe can be computed by the methods already given. If the sum of the flows through the several pipes is known, but the pressure drop is unknown, the problem may best be solved by calling the loss of head between the two points H and writing an expression for the flow in each pipe in terms of H. This gives an equation in which H is the only unknown, which can be readily solved as illustrated in Example I.

EXAMPLE I

Two tanks 5000 ft. apart are connected by two old W.I. pipes, one 12 in., and the other 8 in., in diameter. What must be the difference in elevation to make the combined flow 20 c.f.s. and what will be the velocity in each pipe?

With $\epsilon = 0.002$, f for the 12-in. pipe will be 0.023 and for the 8-in., 0.026. Then for the 12-in. pipe, $H = \dfrac{0.023 \times 5000\, V^2}{64.32}$ $V = \dfrac{8.02\sqrt{H}}{\sqrt{115}} = 0.748\sqrt{H}$

and $Q = 0.7854\,V = 0.587\,\sqrt{H}$. For the 8-in. pipe, $H = \dfrac{0.026 \times 5000\,V^2}{0.667 \times 64.32}$, $V = \dfrac{8.02\,\sqrt{H}}{\sqrt{195}} = 0.574\,\sqrt{H}$, and $Q = 0.349\,V = 0.201\,\sqrt{H}$. Therefore, the total discharge, $20 = 0.789\,\sqrt{H}$, and $H = 640$ ft. The velocity in the 12-in. pipe $= 0.748 \times 25.3 = 18.9$ ft. per sec., and the velocity in the 8-in. pipe $= 0.574 \times 25.3 = 14.5$ ft. per sec.

Another fairly simple case is that of a branching pipe connecting one source of supply and two outlets or two sources of supply and one outlet. This is illustrated by Prob. 36–2. When the elevation of the hydraulic gradient at the junction is unknown, it is probably best to assume a value for this and compute the flows in the main pipe and each branch. If the sum of these latter does not equal the flow in the main pipe, a new assumption is made

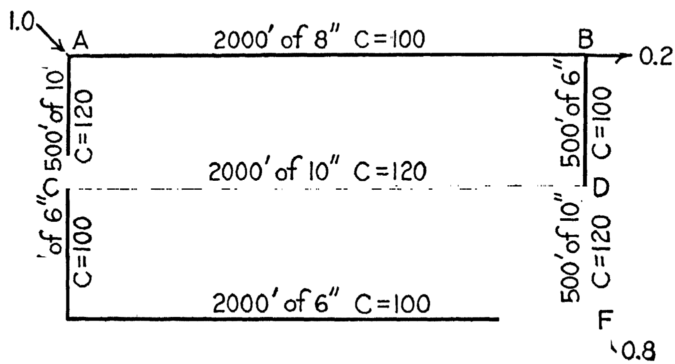


FIG. 76.

and this process is continued until a balance is secured. Or graphs may be plotted of the sum of the flows in the two branches, and of the flow in the main pipe, as functions of the elevation at the junction. The intersection of these lines gives the answer. However, when f is taken to be independent of the flow, the problem may, if desired, be worked by simultaneous equations, the elevation of the hydraulic gradient at the junction and the flow in two branches being the three unknowns.

The general problem of a network of pipes such as arises in city water supply is best worked by the following method due to Professor Gordon M. Fair [15] and based on the Hardy Cross

[15] "Analyzing Flow in Pipe Networks," *Eng. News-Record,* Vol. 120 (Mar. 3, 1938), pp. 342–343.

method of iteration. Figure 76 illustrates a part of a network which might be composed of the water mains around two city blocks. Let Q be the total flow entering at A, and Q_0 be the assumed flow through ABD, and Q_0' be the assumed flow through ACD. Of course, $Q_0 + Q_0'$ must equal Q. Using (30.3) and dropping the subscript f for the rest of this article, we have $h = k_1 L \dfrac{V^n}{D^m}$, or, for any one pipe, $h = k_3 V^n = k\,Q^n$. If the assumed flows Q_0 and Q_0' are in error by an amount Δ_0, we may write

$$Q_1 = Q_0 + \Delta_0 \qquad \text{and} \qquad Q_1' = Q_0' - \Delta_0,$$

the subscript 1 indicating the corrected values. Then the loss of head through ABD is

$$\Sigma\, k\, Q_1^n = \Sigma\, k\, (Q_0 + \Delta_0)^n = \Sigma\, k\, (Q_0^n + n\,\Delta_0 Q_0{}^{n-1} + \ldots),$$

the summation meaning the sum of the values for the various sections of the pipe such as AB and BD. The loss of head through ACD is

$$\Sigma\, k' Q_1'^n = \Sigma\, k'\, (Q_0' - \Delta_0)^n = \Sigma\, k'\, (Q_0'^n - n\,\Delta_0 Q_0'^{n-1} + \ldots)$$

But these two losses must be equal. Also, if Δ_0 is small, higher powers may be neglected and only the terms of the expansion shown need be retained.

Then $\quad \Sigma\, k\, Q_0^n + n\Delta_0 \Sigma\, k\, Q_0^{n-1} = \Sigma\, k' Q_0'^n - n\Delta_0 \Sigma\, k' Q_0'^{n-1} \qquad$ and

$$\Delta_0 = -\frac{\Sigma\, k\, Q_0^n - \Sigma\, k' Q_0'^n}{n\,\Sigma\, k\, Q_0^{n-1} + n\,\Sigma\, k' Q_0'^{n-1}} \cdot \frac{\Sigma\, h_0 - \Sigma\, h_0'}{n\,\Sigma \dfrac{h_0}{Q_0} + n\,\Sigma \dfrac{h_0'}{Q_0'}}, \qquad \text{or}$$

$$(36.1) \qquad \Delta_0 = -\frac{\Sigma\, h_0 \text{ (with attention to direction of flow)}}{n\,\Sigma \dfrac{h_0}{Q_0} \text{ (without reference to direction of flow)}}$$

In equation (36.1) the summation is all the way around the loop. This means that to obtain the correction to be applied to our first assumed flow, we go around the loop in a clockwise direction and add the h's when we are moving with the current and subtract them where we are moving against it. The total is called

$\Sigma\, h_0$. (If our assumptions had been correct, this would of course be zero.) Then the ratios $\frac{h_0}{Q_0}$ for each pipe are added (but without regard to sign) and the sum is called $\Sigma\, \frac{h_0}{Q_0}$. Then $\frac{\Sigma\, h_0}{n\, \Sigma\, \frac{h_0}{Q_0}}$ is the negative of the correction to be applied to each flow. There is one further complication. When any part of the loop is also a part of another loop which is being adjusted at the same time, the correction from the other loop must be applied to this loop as well, but with sign reversed. This is illustrated in Example II.

If pipes are rough and the velocities high, so that equation (29.1) with a constant f can be used, n will be 2. If the Hazen-Williams formula is to be used, $n = 1.852$ (or say 1.85). The proper value for other formulas should be evident after the study of Art. 51.

EXAMPLE II

In Fig. 76, pipes AB, CD, and EF are 2000 ft. long, and AC, CE, BD, and DF are 500 ft. long. AB is an old 8-in. pipe for which $C = 100$, AC, CD, and DF are new 10-in. pipes for which $C = 120$, and BD, CE, and EF are old 6-in. pipes for which $C = 100$. All pipes are cast iron. If a flow of one million gal. per day enters at A, 200,000 gal. per day leaves at B, and 800,000 gal. per day at F, find the flow in each pipe and the elevation of the hydraulic gradient at each lettered point above that at F.

The solution is given in the following table. It was first assumed that of the 1.0 m.g.d. entering at A, 0.3 would go through AB and 0.7 through AC. This would necessitate 0.1 in BD. It was also assumed that at C the flow would divide with 0.6 in CD and 0.1 in CE. These assumptions happen to be nearly correct, so that a solution was obtained after two corrections. If the assumptions had been further from the truth, the final answers would have been the same, but three or four corrections would have been required.

The values in the column headed $1000\, S_0$ were obtained from the graph of the Hazen-Williams formula in Appendix A, but tables, the Hazen-Williams slide rule, or long hand computation would give the same results. The column headed h_0 is obtained by multiplying $1000\, S_0$ by the length in thousand feet. The algebraic sum of h_0 for each loop, divided by 1.85 times the sum of the $\frac{h_0}{Q_0}$ column for the same loop, gives $-\,0.002$ for the upper loop and $+\,0.032$ for the lower. Then for the first three pipes the correction is $+\,0.002$, for CD the correction is $+\,0.002 + 0.032 = +\,0.034$ in the upper loop, and $-\,0.034$ in the lower loop. For the last three pipes the correction is $-\,0.032$. Applying these corrections to the Q_0's gives the Q_1's.

Network				Assumed Flow					First Correction				
Pipe	Length 1000 ft.	Diam. in.	C	Q_0 m.g.d.	$1000\,S_0$	h_0 ft.	$\frac{h_0}{Q_0}$	Δ_0 m.g.d.	Q_1 m.g.d.	$1000\,S_1$	h_1 ft.	$\frac{h_1}{Q_1}$	Δ_1 m.g.d.
	ABDCA												
AB	2.00	8	100	+ 0.3	1.62	+ 3.24	10.80	+ .002	+ .302	1.64	+ 3.28	10.86	− .006
BD	0.50	6	100	+ 0.1	0.86	+ 0.43	4.30	+ .002	+ .102	0.89	+ 0.44	4.31	− .006
AC	0.50	10	120	− 0.7	1.88	− 0.94	1.34	+ .002	− .698	1.86	− 0.93	1.33	− .006
CD	2.00	10	120	− 0.6	1.41	− 2.82	4.70	+ .002 + .032	− .566	1.27	− 2.54	4.48	− .006 − .003
						− 0.09 ÷ 21.14 × 1.85 = − .002					+ 0.25 ÷ 20.98 × 1.85 = +.006		
	CDFEC												
CD	2.00	10	120	+ 0.6	1.41	+ 2.82	4.70	− .032 − .002	+ .566	1.27	+ 2.54	4.48	+ .003 + .006
DF	0.50	10	120	+ 0.7	1.88	+ 0.94	1.34	− .032	+ .668	1.74	+ 0.87	1.30	+ .003
CE	0.50	6	100	− 0.1	0.86	− 0.43	4.30	− .032	− .132	1.43	− 0.72	5.45	+ .003
EF	2.00	6	100	− 0.1	0.86	− 1.72	17.20	− .032	− .132	1.43	− 2.86	21.67	+ .003
						+ 1.61 ÷ 27.54 × 1.85 = + .032					− 0.17 ÷ 32.90 × 1.85 = −.003		

				Second Correction					Results			Elevation	
Pipe	Length 1000 ft.	Diam. in.	C	Q_2 m.g.d.	$1000\,S_2$	h_2 ft.	$\frac{h_2}{Q_2}$	Δ_2 m.g.d.	Q_3 m.g.d.	$1000\,S$	h	Point	h
	ABDCA												
AB	2.00	8	100	+ .296	1.57	+ 3.14	10.61	+ .001	.297	1.57	+ 3.14	*A*	4.41
BD	0.50	6	100	+ .096	0.80	+ 0.40	4.16	+ .001	.097	0.82	+ 0.41	*B*	1.27
AC	0.50	10	120	− .704	1.91	− 0.96	1.36	+ .001	.703	1.91	− 0.95	*C*	3.46
CD	2.00	10	120	− .575	1.30	− 2.60	4.52	+ .001	.574	1.30	− 2.60	*D*	0.86
						− 0.02 ÷ 20.65 × 1.85 = − .001					0.00	*E*	2.77
	CDFEC											*F*	0.00
CD	2.00	10	120	+ .575	1.30	+ 2.60	4.52	.000 − .001	.574	1.30	+ 2.60		
DF	0.50	10	120	+ .671	1.74	+ 0.87	1.30	.000	.671	1.74	+ 0.87		
CE	0.50	6	100	− .129	1.38	− 0.69	5.35	.000	.129	1.38	− 0.69		
EF	2.00	6	100	− .129	1.38	− 2.76	21.40	.000	.129	1.38	− 2.76		
						+ 0.02 ÷ 32.57 × 1.85 = .000					+ 0.02		

Repeating the process gives Q_2. The next repetition gives Q_3. In this case the Δ_3's would all be zero, so that the Q_3's are the final answers. There is still a discrepancy of 0.02 in the h's in the lower loop, and this has been arbitrarily adjusted by changing the $+ 0.87$ to $+ 0.86$ and the $- 2.76$ to $- 2.77$.

PROBLEMS

36-1. In Prob. 35–11, it is desired to increase the flow and to decrease the head required. If a new 16-in. riveted steel pipe is laid parallel to the old 14-in. pipe, what will be the head required for a total flow of 10,000 g.p.m.?
Ans. 29.2 ft.

36-2. A reservoir A, has its water surface at elevation 900 ft. A new cast-iron pipe, 12-in. inside diameter, runs for 10,000 ft. to point B, where the pipe divides. From there a 10-in. new cast-iron pipe runs 5000 ft. to point C at elevation 800 ft., and discharges into the open air. Also from B, an 8-in. new cast-iron pipe runs 10,000 ft. to point D at elevation 700 ft., and discharges into the open air. Find the elevation of the hydraulic gradient at B, and the flow through AB, BC, and BD.

36-3. How many 4-in. pipes placed in parallel will be required to carry the same flow as one 12-in. pipe, with the same pressure drop between the two points? Work by both the Hazen-Williams formula with $C = 120$, and (30.2) with $\epsilon = 0.0005$ ft. *Ans.* Approximately 18 pipes.

36-4. If pipes CE and EF were omitted from the network of Example II, and if the whole million g.p.d. entered at A and left at D, find the flow through each side of the loop, and the loss in head from A to D.
Ans. Loss in head = 5.18 ft.

36-5. Solve Example II if no water leaves at B, but the whole million g.p.d. leaves at F. *Ans.* Total drop from A to F is 5.49 ft.

37. Maximum Power from a Pipe Line.—If a pipe line from an elevated source of supply delivers water to a lower point, the number of horsepower in the water at the lower point, as pointed out in Art. 13, is $\frac{Q\,h}{8.8}$, where h is the sum of the pressure and velocity heads at the lower point. But the power does not vary directly as Q, because h is also a function of Q. When the flow is throttled down to a very small amount, h approaches its maximum value of H, the vertical distance from the discharge point to the free surface at the supply. As the flow is made larger, the loss of head in the pipe becomes larger and h becomes less, but the product of $Q\,h$ increases. At a certain flow it becomes a maximum and for still larger flows it again becomes less. The limit is reached when all but one velocity head is lost in the pipe. This is illustrated in the following examples and problems.

EXAMPLE I

A concrete lined tunnel ($\epsilon = 0.002$), 10 ft. inside diameter and two miles long, carries water from a reservoir to a point on a river where the water surface is 600 ft. below the surface of the reservoir. The mouth of the tunnel is submerged. What will be the flow, and what power would remain in the water at discharge? Considered as a transmitter of power, what would be the efficiency of the tunnel?

From Fig. 65, $f = 0.014$, $600 = \dfrac{0.014 \times 10{,}560}{10} h_v + h_v$; $h_v = \dfrac{600}{15.78}$ $= 38.02$ ft. Then $V = 8.02\sqrt{38.02} = 49.45$ ft. per sec. and $Q = 78.54 \times 49.45 = 3880$ c.f.s. The total head at exit is simply the velocity head, 38.02 ft., therefore the horsepower is $\dfrac{3880 \times 38.02}{8.8} = 16{,}800$ hp. Of the total head of 600 ft., only 38.0 ft. is transmitted, so that the efficiency of the tunnel as a transmitter of power is $\dfrac{38}{600}$ or 6.33 per cent.

EXAMPLE II

If the flow in Example I were reduced by a gate, or by a constriction at its lower end, until only 200 ft. were lost in the pipe, what would be the efficiency as a transmitter of power, the discharge, and the horsepower?

The efficiency would be $\dfrac{100\,(600 - 200)}{600} = 66.7$ per cent. $200 = \dfrac{0.014 \times 10{,}560}{10} h_v$, $h_v = \dfrac{200}{14.78} = 13.53$. $V = 8.02\sqrt{13.53} = 29.49$ ft. per sec. $Q = 78.54 \times 29.49 = 2317$ c.f.s. Power $= \dfrac{2317 \times 400}{8.8}$ $= 105{,}300$ hp.

PROBLEMS

37–1. Find the flow that would use 400 ft. of head as loss in the pipe of the above examples. What would be the horsepower output?

Ans. 3280 c.f.s.; 74,500 hp.

37–2. If it were possible to reduce the pressure at the lower end to absolute zero, and the water barometer were 33.9 ft., what would be the discharge and horsepower? *Ans.* 3990 c.f.s.; 2840 hp.

37–3. What loss of head in the pipe would make the pipe line efficiency in the above problems 95 per cent? What would be the discharge and power?

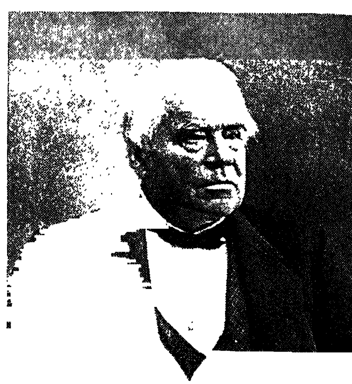
Ans. 30 ft.; 898 c.f.s.; 58,100 hp.

37–4. Plot the results of the examples and Probs. 37–1 to 37–3 as curves with discharge as abscissas and gross horsepower, total head, velocity head and pressure head as ordinates.

LEONHARD EULER (1707–1783)—Swiss mathematician. Developed the fundamental differential equations of fluid flow.

ANTOINE DE CHEZY (1718–1798)—French engineer, who was perhaps the first to study "losses" quantitatively.

GIOVANNI BATTISTA VENTURI (1746–1822)—Italian priest and teacher. Studied recovery of head in expanding tubes, and finally explained all difficulties in Bernoulli's theorem.



JAMES B. FRANCIS (1815–1892)—American hydraulic engineer. Measured flow over weirs, deduced weir formulas, and developed the first satisfactory turbine.

We can now generalize the preceding as follows: If H is the total available head, the net head will be $H - h_f$ and the power output will be $\frac{Q\,(H - h_f)\,e}{8.8}$. But $h_f = \frac{f\,L\,V^2}{D\,2\,g}$ and $Q = A\,V$. Therefore $P = \frac{e\,A\,V}{8.8}\left(H - \frac{f\,L\,V^2}{D\,2\,g}\right)$. With D, A, L, and e constant, P is a maximum when $\frac{dP}{dV} = 0$ or $\frac{e\,A}{8.8}\left(H - \frac{3\,f\,L\,V^2}{D\,2\,g}\right) = 0$ or $H = 3\,h_f$, and

$$h_f = \frac{H}{3} \tag{37.1}$$

Therefore, Example II gave the maximum power possible from that pipe and total head. In actual power plants, however, the supply line is practically never operated for maximum power, because it makes the power obtained from a given flow of water too expensive. The only case where this would be the proper thing to do would be when the demand for power was limited, but the supply of water was ample at all times. Generally it is financially advisable to build a larger pipe or produce less power, so that the efficiency of the pipe line will be 90 per cent or more. This is discussed in the next article.

PROBLEMS

37–5. A power plant has an available head of 300 ft. and the pipe line is 5000 ft. long and 4 ft. in diameter. If $f = 0.012$ and the supply of water is unlimited, what is the maximum gross power (neglecting all losses except pipe friction), that can be delivered to the power plant? *Ans.* 5920 hp.

37–6. A 12-in. pipe 2000 ft. long, brings water from a reservoir to a Pelton water wheel 150 ft. below the surface of the reservoir. The nozzle is designed so that the pipe is transmitting the maximum possible horsepower. $f = 0.016$. The combined efficiency of nozzle and wheel is 80 per cent. What is the horsepower output? *Ans.* 71.6 hp.

37–7. The pipe from a mountain reservoir to a Pelton water wheel is 5000 ft. long and 2 ft. in diameter. The water surface is 600 ft. above the nozzle. Find the velocity in the pipe for maximum power if $f = 0.020$. (Neglect minor losses, as they are assumed to be included with the pipe friction.) What would be the necessary area of the jet to give this condition, and what would be the horsepower of the jet?

Ans. 16.04 ft./sec.; 0.314 sq. ft.; 2290 hp.

37–8. Find the horsepower delivered per c.f.s. in Examples I and II and Probs. 37–1 and 37–3. Does this show which is the best flow to use?

Ans. 4.33; 45.4; 22.7; 64.8.

38. Economic Diameter.—When the flow to be carried is fixed either by the supply or by the demand, the economic diameter of pipe to use can be computed. One method of doing this is to compute the total annual cost of pipe plus that of the power used in overcoming pipe friction for several diameters, and to plot a curve of cost against diameter. The low point on the curve determines the most economical diameter. Where the costs vary with the diameter according to no simple law, that is about the only way of solving the problem. However, for preliminary calculations at least, it is allowable to make simplifying assumptions, which make it possible to work the problem by calculus as illustrated in the following example.

EXAMPLE I

Find the economic diameter of a steel pipe which will be carrying 400 c.f.s. one-sixth of the time, 300 c.f.s. one-sixth of the time, 200 c.f.s. one-sixth of the time, and 100 c.f.s. one-half of the time. Assume the cost of steel in place as 10 cents per lb., with 10 per cent extra for laps, rivets, etc.; the shell to be 5/16 in. thick; $f = 0.017$; the efficiency of the turbine to be 85 per cent; the power to be worth 3 mills per kilowatt hour, and the annual cost to be 7 per cent of the original cost.

Steel weighs 490 lb. per cu. ft., and 5/16 in. = 0.0260 ft.; therefore, the cost per ft. of pipe is $1.1 \times 0.10 \times 490 \times 0.026 \times \pi D = 4.402\ D$ dollars, with D in ft. Then the annual cost per ft. of pipe is $0.07 \times 4.402\ D = 0.308\ D$ dollars. Three mills per k.w. hour = 7.2 cents per k.w. day = \$26.28 per k.w. year = \$19.60 per horsepower year. This is for power as sold. Since the efficiency is only 85 per cent, the value of gross power is only $0.85 \times 19.60 = \$16.66$ per horsepower year. The head lost in friction by (35.1) is $\dfrac{0.017\ Q^2}{39.7\ D^5}$. Therefore, the power lost in friction $= \dfrac{h_f Q}{8.8}$ $= \dfrac{0.017\ Q^3}{39.7 \times 8.8\ D^5} = \dfrac{0.0000487\ Q^3}{D^5}$. Then the annual value of this power is $16.66 \times 0.0000487\ Q^3 D^{-5} = 0.000811\ Q^3 D^{-5}$ dollars.

If we represent the total annual cost by J, this gives the equation $J = 0.308\ D + 0.000811\ Q^3 D^{-5}$. If Q is independent of D, as it is here, the minimum cost will be when $\dfrac{dJ}{dD} = 0$; that is, when $0.308 = 5 \times 0.000811\ Q^3 D^{-6}$, or $D^6 = 0.01317\ Q^3$. The equivalent uniform flow that would give the same power loss as the variable flow here given, will be the cube root of the average of the cubes of the Q's. $100^3 = 1{,}000{,}000$; $200^3 = 8{,}000{,}000$; $300^3 = 27{,}000{,}000$; and $400^3 = 64{,}000{,}000$. Since $\dfrac{8 + 27 + 64}{3} = 33$, and $\dfrac{33 + 1}{2} = 17$, the average of the cubes of the Q's is 17,000,000, and the equivalent Q is 257 c.f.s. Then $D^6 = 0.01317 \times 17{,}000{,}000 = 224{,}000$, $D^2 = 60.7$, and $D = 7.79$ ft.

If the annual cost per foot of pipe is represented by C_1D, and the annual cost of the power lost per foot of pipe be $C_2Q^3D^{-5}$, by the same method as above we find that the economical diameter is D, when

$$D^6 = \frac{5\,C_2Q^3}{C_1} \tag{38.1}$$

If the tensile strength in the gross section of the pipe wall is S_t pounds per square inch, the steel thickness is t_w inches, and the head on the pipe is H feet, by taking as a free body a longitudinal half of a foot of length of pipe and the water in it, we see that

$$w\,H\,D = 12\,t_wS_t \times 2 = 24\,t_wS_t \tag{38.2}$$

Taking S_t as 12,000 pounds per square inch (say 15,000 pounds per square inch in the net section of the riveted joint, and 80 per cent joint efficiency), this gives $H = 24 \times \frac{5}{16} \times \frac{12{,}000}{62.5 \times 7.79}$ $= 185$ feet. That is, this pipe would be good up to a head of 185 feet. For any greater head, the thickness would have to be increased and t_w becomes a function of the diameter. Then the volume of pipe, $\frac{\pi\,D\,t_w}{144}$ cubic feet per foot of pipe, becomes $\frac{\pi\,D}{144} \times \frac{w\,H\,D}{24\,S_t}$ $\frac{\pi\,w\,H\,D^2}{3454\,S_t}$ and the annual cost varies as the square of the diameter. Then

$$J = C_3D^2 + C_2Q^3D^{-5} \tag{38.3}$$

and placing $\frac{dJ}{dD}$ equal to zero,

$$D^7 = \frac{5\,C_2Q^3}{2\,C_3} \tag{38.4}$$

This is the rule for finding the economic diameter when the head and flow are large enough so that the shell thickness will depend on the diameter. For lower heads or flows, where the shell thickness is determined only by such practical considerations as stiffness in handling and provision for corrosion, equation (38.1) should be used.

EXAMPLE II

Solve Example I if the maximum head on the pipe is 400 ft., and $S_t = 12{,}000$ lb. per sq. in.

If the pipe shell is t_w in. thick, $2\,t_w \times 12{,}000 \times 12 = 62.5 \times 400\,D$ and $t_w = \dfrac{25{,}000\,D}{288{,}000} = 0.0868\,D$ in. $= 0.00723\,D$ ft. Then the cost per ft. of pipe will be $0.11 \times 490 \times 0.00723\,\pi\,D^2 = 1.225\,D^2$ dollars. Then the annual cost $= 0.07 \times 1.225\,D^2 = 0.0857\,D^2$ dollars, and $C_3 = 0.0857$. Then by (38.4), $D^7 = \dfrac{5 \times 0.000811 \times 17{,}000{,}000}{2 \times 0.0857} = 402{,}000$ and $D = 6.32$ ft. From this $t_w = 0.0868\,D = 0.549$ in. and a 9/16-in. shell would be required. The difference between 0.549 and the commercial size, 9/16, might require a slight further correction in the economic diameter, but this is as far as we shall carry the matter here. The curve of cost is fairly flat near its minimum point, so that a change of a few per cent in the diameter adopted makes very little difference. The fact that we cannot predict with any certainty the value of power throughout the life of the pipe line, or the interest rate, and many other uncertainties, make the actual selection of the best size open to debate. The numerical values used in Examples I and II were taken from a recent paper on the subject.[16] In the writer's opinion 0.07 is too low for the ratio of annual cost to total cost. It consists of interest, including interest during construction and development of market, depreciation, obsolescence, and taxes in so far as these are based on original cost, and will generally run from 0.10 to 0.12. Also, if the 3 mills per kilowatt hour is the bus bar price, the efficiency should be the average over-all efficiency of the whole plant, which would rarely if ever be as high as 85 per cent, and would generally be between 75 and 80.

The problem of the economic diameter of a pipe through which liquid is to be pumped, is of the same sort as those just considered, the only difference being that the efficiency of the pump appears in the denominator instead of in the numerator, for if 100 horsepower is lost in the pipe, and the pump efficiency is 80 per cent, 125 horsepower will have to be supplied to the pump to overcome the loss.

Since f depends on the pipe size, in the actual problem it will generally have to be revised after a first solution for D has been found.

[16] "Economic Diameter of Steel Penstocks," by Charles Voetsch and M. H. Fresen, *Trans. A.S.C.E.*, Vol. 103 (1938), pp. 89–132. They found 7.40 and 6.03 for the two cases, but their variation in Q was not quite the same as ours.

PROBLEMS

38-1. For a minimum thickness of 3/8 in., develop a rule as to when equation (38.1) applies, and when (38.4). The fluid is to be water and the maximum tension in the gross section of the pipe is to be limited to 12,000 lb. per sq. in.

Ans. When $H\,D < 1728$ use (38.1).
When $H\,D > 1728$ use (38.4).

38-2. Solve Example II with only the following data changed. The flow is 800 c.f.s. (continuously), and the annual cost is 10 per cent of the original cost. *Ans.* 9.77 ft.

38-3. In the preceding problem, what is the necessary thickness of pipe shell; what is the value of the energy lost per ft. per year; what is the annual cost of the pipe per ft.; and what is the ratio of the last two?

Ans. 0.847 or 7/8 in.; $4.67; $11.69; 0.40

38-4. Show that the last answer will always be so if the head is high enough to control the thickness. This is called the Adams principle in honor of A. L. Adams,[17] who first discovered it without the use of calculus.

38-5. Gasoline is to be pumped from one tank to another at approximately the same level through old W.I. pipe at the rate of 1 c.f.s., for 8 hr. each day. The pump efficiency is 74.6 per cent; power costs 2 cents per k.w.hr.; annual cost is 12 per cent of first cost; and the other data are as in Example I. Find the economical diameter. *Ans.* Use 8-in. pipe.

38-6. Representing the data of the example by letters, go through the proof of (38.1) and (38.4) and give a simple definition of C_1, C_2, and C_3.

39. Velocity Distribution.—In most of what has gone before, we have assumed that the velocity was uniformly distributed throughout the cross-section. This is nearly true in the jet from a sharp-edged orifice and is approximately true in the jet from a

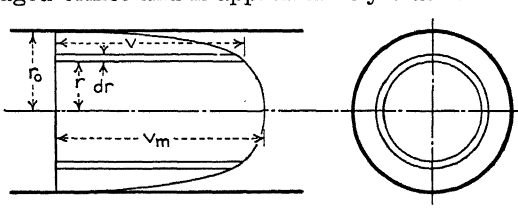


FIG. 77.—Diagram of velocity distribution in pipe.

nozzle and in a pipe near a rounded entrance. But as the flow proceeds down the pipe, the drag of the walls slows up the outer layers of the liquid, while, by the law of continuity, the inner layers must speed up to keep the average the same. This unequal distribution necessitates a correction to some of the things we have said about kinetic energy and momentum.

[17] *Trans. A.S.C.E.*, Vol. 59 (Dec., 1907), pp. 173–194.

In Fig. 77, r_o is the radius of the pipe and r is any other smaller radius to some point in the liquid where the velocity is v. At the center where $r = 0$ we have the maximum velocity, v_m. The mean velocity is V. Now consider the flow for one second through an elementary ring with inner radius r and outer radius $r + dr$. This is $dQ = v \times 2\,\pi\,r\,dr = v\,dA$. Its mass is $\frac{w}{g}\,v\,dA$ and its kinetic energy is $\frac{w}{2\,g}\,v^3\,dA$. Then the total kinetic energy is $\frac{w}{2\,g}\int v^3\,dA$. Figured by the ordinary method, the total kinetic energy is $\frac{w\,Q\,V^2}{2\,g} = \frac{w}{2\,g}\,A\,V^3$. Call the ratio of these two values α, then

$$\alpha \quad \frac{\int v^3\,dA}{V^3\,A} \tag{39.1}$$

It can be shown that this will always be more than one except when v is constant, when it is obviously one.

The total kinetic energy passing any given section per second is $\frac{w}{2\,g}\int v^3\,dA = \frac{w}{2\,g}\,\alpha\,V^3A = \frac{w}{2\,g}\,\alpha\,Q\,V^2$. Therefore the kinetic energy per pound of fluid is $\alpha\,\frac{V^2}{2\,g}$, so that the true formula for velocity head in terms of average velocity is

$$h_v = \alpha\,\frac{V^2}{2\,g} \tag{39.2}$$

Rehbock has shown that approximately

$$\alpha = 1 + \delta^2 \tag{39.3}$$

where δ (delta) is the ratio by which v_m exceeds V; that is,

$$v_m = (1 + \delta)\,V \tag{39.4}$$

In rough pipes at high velocities, Nikuradse found the data given in the first two columns of Table II. The last two columns have been computed by the writer from (39.4) and (39.3).

TABLE II.—VALUES OF α FOR ROUGH PIPES

$\frac{r_0}{\epsilon}$	$\frac{V}{v_m}$	δ	α
15	0.755	0.324	1.11
30.6	0.781	0.280	1.08
60	0.800	0.250	1.06
126	0.818	0.222	1.05
252	0.833	0.200	1.04
507	0.845	0.183	1.03

These values of α are comparable to those observed in the Sudbury Aqueduct and in other engineering tests.[18] At lower velocities and in smooth pipes and in open channels, the values of α are, in general, larger, as will be noted in Chapters V and VI.

In a good nozzle α is about 1.01, so that when the apparent efficiency of a nozzle is 95 per cent, its true efficiency is about 96 per cent. If the jet is used for power in a Pelton water wheel the whole $\alpha \frac{V^2}{2g}$ foot pounds of energy per pound of water is available for use. On the other hand, at exit from a pipe line the whole $\alpha \frac{V^2}{2g}$ is lost. The omission of the α in our work up to this point has been an approximation sometimes warranted and sometimes not. It is, however, typical of American engineering practice, which has in general entirely ignored this point (and to a large extent been ignorant of it).

A similar point is involved in momentum computations. In Fig. 77, the momentum of dQ is $\frac{w}{g} v^2\, dA$ and the whole momentum per second is $\frac{w}{g}\int v^2\, dA$. Figured by the ordinary method the momentum is $\frac{W}{g} V = \frac{w}{g} Q\, V = \frac{w}{g} V^2A$. Writing $\frac{\int v^2\, dA}{V^2A} = \beta$, we have

$$\text{Momentum per cubic foot} = \beta \frac{w}{g} V \tag{39.5}$$

[18] See *Applied Fluid Mechanics* by O'Brien and Hickox, McGraw-Hill Book Co. (1937), p. 272.

Approximately,

$$\beta = 1 + \frac{\delta^2}{3} \tag{39.6}$$

Therefore the error in momentum is only one-third the error in kinetic energy, and can usually be ignored.

EXAMPLE

It was formerly often stated that the velocity distribution for turbulent flow in a pipe could be represented as a cylinder of length $\frac{v_m}{2}$ plus half an ellipsoid of revolution; that is, that $v = \frac{v_m}{2}\left(1 + \sqrt{1 - \frac{r^2}{r_0^2}}\right)\cdot$ Find α, β, and δ if this were true and check the approximate rules (39.3) and (39.6).

The necessary integrations are best made by substituting $r = r_0 \sin\theta$ Then $\sqrt{1 - \frac{r^2}{r_0^2}} = \cos\theta,\quad dr = r_0\cos\theta\, d\theta,\quad$ and $\quad v = \frac{v_m}{2}(1 + \cos\theta)$.

$$Q = \int v\, dA = 2\,\pi \int v\, r\, dr = \pi\, v_m r_0^2 \int (1 + \cos\theta)\sin\theta\cos\theta\, d\theta$$

$$= A\, v_m \int_0^{\pi/2} \sin\theta\, d\,(\sin\theta) + A\, v_m \int_{\pi/2}^{0} \cos^2\theta\, d\,(\cos\theta) = \frac{A\, v_m}{2} + \frac{A\, v_m}{3}$$

$$= \frac{5}{6} A\, v_m.$$

Therefore $V = \frac{5}{6} v_m$, $v_m = \frac{6}{5} V$, and $\delta = 0.20$.

The momentum per sec.

$$= \frac{w}{g}\int v^2\, dA = \frac{2\,\pi\, w}{g}\int_0^{r_0} v^2 r\, dr = \frac{\pi\, w\, v_m^2 r_0^2}{2\, g}\int_0^{\pi/2} (1 + \cos\theta)^2 \sin\theta\cos\theta\, d\theta$$

$$= \frac{w\, A\, v_m^2}{2\, g}\left\{\int_0^{\pi/2} \sin\theta\, d\,(\sin\theta) + 2\int_{\pi/2}^{0} \cos^2\theta\, d\,(\cos\theta) + \int_{\pi/2}^{0} \cos^3\theta\, d\,(\cos\theta)\right.$$

$$\left.\frac{w\, A\, v_\pi^2}{2\, g}\right\{\frac{1}{2} + \frac{2}{3} + \frac{1}{4}\left.\right\} = \frac{17\, w\, A\, v_m^2}{24\, g} = \beta\,\frac{w}{g} A\, V^2$$

Then $\beta = \frac{17}{24} \times \frac{v_m}{V^2} = \frac{17}{24} \times \frac{36}{25} = \frac{51}{50} = 1.02$. The approximate rule gives $\beta = 1 + \frac{0.04}{3} = 1.013$.

The kinetic energy per sec. $= \frac{w}{2\, g}\int v^3\, dA = \frac{\pi\, w}{g}\int_0^{r_0} v^3 r\, dr$

$$= \frac{\pi\, w\, v_m^3}{8\,g} \int_0^{\pi/2} (1 + \cos\theta)^3 \sin\theta \cos\theta \, d\theta$$

$$= \frac{w\,A\,v_m^3}{8g} \left\{ \int_0^{\pi/2} \sin\theta \, d\,(\sin\theta) + 3 \int_{\pi/2}^{0} \cos^2\theta \, d\,(\cos\theta) \right.$$

$$\left. + 3 \int_{\pi/2}^{0} \cos^3\theta \, d\,(\cos\theta) + \int_{\pi/2}^{0} \cos^4\theta \, d\,(\cos\theta) \right\}$$

$$= \frac{w\,A\,v_m^3}{8\,g} \left(\frac{1}{2} + 1 + \frac{3}{4} + \frac{1}{5}\right) = \frac{49\,w\,A\,v_m^3}{160\,g} = \alpha \frac{w}{2\,g} A\,V^3.$$

Then $\alpha = \dfrac{49\,v_m^3}{80\,V^3} = \dfrac{49}{80} \times \dfrac{216}{125} = 1.058$, or, say, 1.06. The approximate rule gives $\alpha = 1.04$. The failure of the rules to give closer answers in this case should not be charged against them, as actual flow does not distribute itself in quite this way.

PROBLEMS

39-1. Solve the Example if $v = v_m \left(1 - \dfrac{r^2}{r_0^2}\right)$.

Ans. $\alpha = 2$; $\beta = 1.33$; $\delta = 1$.

39-2. Solve the Example if $v = v_m \left(1 - \dfrac{0.3\,r^2}{r_0^2}\right)$.

Ans. $\alpha = 1.031$; $\beta = 1.010$; $\delta = 0.176$

39-3. In one run on his roughest pipe, Nikuradse found the following data:

$\frac{r}{r_0}$	v Cms./Sec.	$\frac{r}{r_0}$	v Cms./Sec.
0.00	76.0	0.50	67.7
0.10	74.8	0.60	64.7
0.20	73.8	0.70	61.0
0.30	72.0	0.80	56.2
0.40	70.0	0.90	48.5
	Approximately	1.00	17.0

By considering the velocity in each ring of thickness equal to one-tenth the radius as the average of the velocities at the two sides of the ring, compute V, δ, α, and β and compare with the approximate formula.

Ans. $V = 57.2$ cms./sec.; $\delta = 0.33$; $\alpha = 1.16$; $\beta = 1.05$

40. Water Hammer.—Let a liquid be flowing from a supply tank through a pipe as shown in Fig. 78. Then suppose that a valve at the outlet of the pipe G is suddenly closed. Since the velocity at this point is reduced to zero, the ordinary law of continuity would require that the velocity at all other points

along the pipe would be brought to zero simultaneously. Actually, this is impossible. Instead, the liquid in the portion of the pipe OP continues to flow at its original velocity, and the liquid in PG is compressed and the pipe stretched to a greater diameter, as is shown to an exaggerated scale in the figure. P is a movable point,

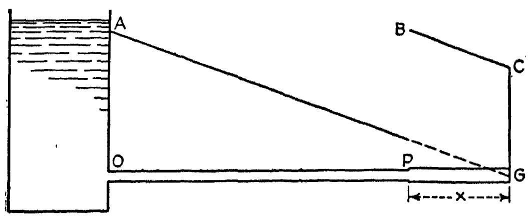


FIG. 78.—Diagram of water hammer in pipe.

starting at G and moving along the pipe to O. At the instant P reaches O, all the liquid in the pipe is under the higher pressure and begins to flow back into the tank. Point P begins to move back toward G, the liquid in OP now moving toward the tank, and that in PG still being at rest. When P reaches G, all the liquid is in motion toward the left, so that the pressure at G will fall below its initial value, and a wave of rarefaction will start back toward the tank. When this reaches O, liquid will start back into the pipe. This series of alternate pressure and rarefaction waves would follow one another indefinitely if it were not for viscosity and certain other minor effects which cause damping, so that equilibrium is finally obtained. This sort of phenomenon is given the name *water hammer*, from the hammering sound caused by the sudden closure of a valve in a long water pipe. In order to get just the effects described above, the valve would have to be closed instantly. This is the impossible limiting case which is approached as the time of closure is made smaller and smaller in comparison with the time required for the wave to travel the length of the pipe and back.

In Physics textbooks it is shown that the *celerity* of travel of compression or rarefaction waves relative to the liquid in which they are traveling, is

$$c = \sqrt{\frac{E_v}{\rho}} = \sqrt{\frac{g}{w}}, \tag{40.1}$$

where E_v is the bulk modulus of elasticity of the liquid. But this equation neglects the effect of the expansion of the pipe (as well as other minor effects). The actual celerity of the wave is

$$c = \sqrt{\frac{K\,g}{w}} \tag{40.2}$$

where K is the apparent bulk modulus of the liquid as modified by the expansion of the pipe. If the pressure at any point in the pipe increases from p_1 to $(1 + \delta)p_1$, the increase in hoop stress in the pipe wall will be $\frac{\delta\, p_1 D}{2\, t_w}$, where t_w is the thickness of the pipe wall, and if E is the modulus of elasticity of the pipe wall, the relative elongation of the circumference will be $\frac{\delta\, p_1 D}{2\, t_w E}$. The proportional increase in the area for small increases in the circumference is twice the latter, or $\frac{\delta\, p_1 D}{t_w E}$. Therefore, the increase in volume per foot of pipe is $\frac{\pi\, \delta\, p_1 D^3}{4\, t_w E}$. But even without the stretching of the pipe, the increased pressure would compress more liquid into the same volume. This extra amount, measured in cubic feet at the original pressure, would be $\frac{\pi\, D^2 \delta\, p_1}{4\, E_v}$. Adding the two expressions gives $\frac{\pi\, D^2}{4}\,\delta\, p_1\left(\frac{1}{E_v} + \frac{D}{t_w E}\right)$. This change in volume must, by definition of bulk modulus, equal the original volume times the change in pressure divided by the equivalent bulk modulus; that is, $\frac{\pi\, D^2 \delta\, p_1}{4\, K}$. Therefore,

$$\frac{1}{K} = \frac{1}{E_v} + \frac{D}{t_w E} \tag{40.3}$$

E for steel is about 29,000,000 pounds per square inch, and E_v for water varies from 300,000 to 320,000 pounds per square inch for ordinary temperatures and pressures. (See Appendix A.) It must be noted that if K is found in pounds per square inch, it must be converted into pounds per square foot before substituting in (40.2) if c is to be obtained in feet per second. Also, we

have assumed the pipe anchored against lengthwise motion. If it expands lengthwise with increases in pressure, K will be smaller than given by (40.3). In fact, the only sure method of obtaining the celerity of compression waves in a pipe is by experiment.

If the length of the pipe is L, the time for P to travel from G to O will be $\frac{L}{c}\cdot$ During this time liquid will continue to flow in at G at the original velocity V; that is, a total of $\frac{\pi D^2 V L}{4 c}$ cubic feet will flow in. The proportional increase in quantity of liquid in the pipe will then be $\frac{\pi D^2 V L}{4 c} \quad \frac{\pi D^2 L}{4} = \frac{V}{c}\cdot$ But by definition of bulk modulus, $K = \frac{\Delta p}{\frac{V}{c}} = \frac{\Delta p c}{V}$, where Δp is the increase in pressure. Therefore $\Delta p = \frac{V K}{c}\cdot$ But $c = \sqrt{\frac{K g}{w}}$; therefore, $K = \frac{c^2 w}{g}$ and

$$\Delta p = \frac{V}{c} \times \frac{c^2 w}{g} = \frac{V c w}{g} \tag{40.4}$$

This is called Joukovsky's law,[19] having been derived by him in 1898. It should be noted that if viscosity effects are neglected, the same Δp will be felt at all points along the pipe, and if, as in Fig. 78, the hydraulic gradient was originally AG, an instantaneous closure of valve G would raise the pressure at G to C, at P to B, etc., line BC being above AG a distance $\frac{\Delta p}{w}\cdot$ There will be a similar line below AG for the wave of rarefaction.

Actually, of course, some time is required to close any valve, but if the time is less than $\frac{2 L}{c}$ seconds, the maximum pressure rise caused is practically the same as for instantaneous closure. The difference is that at any one point, as P, the increase will

[19] For a summary (in English) of Joukovsky's work, see "Water Hammer," by Miss O. Simin, *Proc. Amer. Water Works Assn.*, 1904, pp. 341–424.

not take place instantly, but in a time approximately equal to the time of closing the valve.

If the time of closure is greater than $\frac{2L}{c}$, the maximum pressure rise is reduced. Therefore, if large pressure rises are to be avoided the time of valve closure should be several times $\frac{2L}{c}$. No attempt will be made here to enter further into this complicated question.[20] Besides endeavoring to keep all changes in velocity gradual, water hammer may be protected against by air chambers near the valve, by surge chambers, and by relief valves. In the *hydraulic ram*, water hammer is put to practical use as a means of raising water to higher levels.

EXAMPLE

A so-called 1/8-in. standard steel pipe, has an inside diameter of 0.269 in. and an outside diameter of 0.405 in. For a length of 100 ft., what is the maximum time of valve closure which will produce the maximum increase in pressure and what is the increase if the original velocity was 5 ft. per sec.? The liquid is water at 68° F.

From Appendix A, E_v = 320,000 lb. per sq. in., and $\frac{1}{E_v} = 0.000003125$ sq. in. per lb. $t_w = \frac{1}{2}(0.405 - 0.269) = 0.068$ in., and $\frac{D}{t_w E} = \frac{0.269}{0.068 \times 29{,}000{,}000} = 0.000000136$ sq. in. per lb. Therefore, $\frac{1}{K} = 0.000003125 + 0.000000136 = 0.000003261$, and $K =$,700 lb. per sq. in. = 44,160,000 lb. per sq. ft. Therefore, $\sqrt{\frac{Kg}{w}} = \sqrt{\frac{44{,}160{,}000 \times 32.16}{62.31}} = 4741$ ft. per sec. This is an unusually rapid rate of travel, and is due to the fact that the pipe walls were relatively very thick and the water temperature high. $\frac{2L}{c} = \frac{200}{4741} = 0.0422$ sec. is the maximum time of closure to cause the maximum pressure increase. The latter is $\Delta p = \frac{5 \times 4741 \times 62.31}{32.16} = 45{,}930$ lb. per sq. ft. = 319 lb. per sq. in.

[20] For a brief discussion see Arts. 155 and 156 of R. L. Daugherty's *Hydraulics*, 4th Ed. (1937), McGraw-Hill Book Co. For a more complete treatment, see *Symposium on Water Hammer*, A.S.M.E., 1933. The most recent work is briefly described in *Engineering News-Record*, June 22, 1939. p. 67.

PROBLEMS

40–1. Find the celerity of a compression wave in water at 32° F. contained in a standard 12-in. steel pipe (inside diameter 12.000 in., outside diameter 12.750 in.). *Ans.* 4050 ft./sec.

40–2. Work the preceding problem if the pipe is changed to cast iron with inside diameter 10 in. and outside diameter 11 in. Take Young's modulus for cast iron as 15,000,000 lb. per sq. in. *Ans.* 3950 ft./sec.

40–3. Find the celerity of a compression wave in water at 68° F. contained in a standard 1-in. steel pipe (inside diameter 1.049 in., outside diameter 1.315 in.). *Ans.* 4680 ft./sec.

40–4. Solve the preceding problem if the temperature of the water is changed to 32° F. *Ans.* 4350 ft./sec.

40–5. What would be the maximum time of closing a valve at the end of 10,000 feet of pipe for the conditions of Prob. 40–1? *Ans.* 4.94 sec.

40–6. If the initial velocity in the preceding problem were 5 ft. per sec., what would be the increase in pressure caused by a sudden closure of the valve? *Ans.* 273 lb./sq. in.

40–7. State clearly the definition of bulk modulus of elasticity and the equation for hoop stress in terms of internal pressure, and from these derive (40.3) and (40.4) clearly and in detail.

40–8. Derive (40.4) by equating the loss of kinetic energy of the water stopped to the gain in potential energy of water and pipe.

CHAPTER V

FLOW IN OPEN CHANNELS

41. Terms Used.—A canal or river is called an *open channel* because the liquid has a free surface. Flow in a pipe or other conduit where the liquid does not fill the whole conduit but has a free surface is also open channel flow. Flow in open channels is in many ways similar to the flow in pipes, with one important distinction. In pipes the hydraulic gradient may be far above the pipe or even below it, but in open channels the hydraulic gradient *is* the free surface. The perimeter of the cross-section exclusive of the free surface is called the *wetted perimeter*. We are interested in this because from one point of view it is the drag on this boundary that makes it necessary for the free surface to slope in the direction of the flow. The air along the free surface also causes some resistance (unless there is a downstream wind just equal to the surface velocity of the liquid), but this resistance is very slight since the bottom layer of air will be carried along by the liquid and the only resisting force is the shearing stress in the air which will never be large.

Imagine a liquid flowing in a uniform channel with its free surface parallel to the bottom of the channel so that the cross-sectional area, A, is constant. Let P_w be the wetted perimeter and S be the longitudinal slope of the bottom and of the free surface. This is to be so small that S may be considered as either the tangent of the angle with the horizontal or as its sine. Let τ_0 be the average shearing stress in the liquid where it comes in contact with the wetted perimeter. Then taking the liquid in one foot of channel as the free body, the component of gravity down the plane must balance the backward shearing force. (Since A was to be constant, the law of continuity requires V to be constant, and therefore there is no accelerating force.) Then

$$(41.1) \qquad w\,A\,S = \tau_0 P_w \qquad \text{or} \qquad \tau_0 = \frac{w\,S\,A}{}$$

The ratio of area to wetted perimeter is thus seen to be fundamental to the study of channel flow. It is called the *hydraulic radius* and is represented by the letter R.

$$R = \frac{A}{P_w} \tag{41.2}$$

In wide, shallow channels like most rivers, the wetted perimeter is very little more than the top width, and area divided by width equals depth; so that the hydraulic radius is sometimes called the *hydraulic mean depth.* The other extreme is a circular pipe flowing very nearly full. Here A is practically $\frac{\pi D^2}{4}$, and P_w is a little less than πD, so that R will be only slightly more than $\frac{D}{4}$.

In order that the student may acquire facility in determining the hydraulic radius of various cross-sections, the following problems are given.

EXAMPLE I

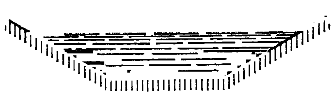


Fig. 79. — Trapezoidal channel section.

In the trapezoidal channel whose cross-section is shown in Fig. 79, the base is 12 ft. and the side slopes are 2 : 1 (two horizontal to one vertical; also written 1 on 2). Find the hydraulic radius when flowing 5 ft. deep.

$A = 5\ (12 + 10) = 110$ sq. ft. The side slopes are each $5\sqrt{5}$, and the wetted perimeter is $12 + 10\sqrt{5} = 12 + 22.36 = 34.36$ ft. Therefore $R = 110 \div 34.36 = 3.20$ ft.

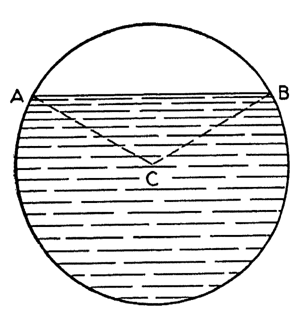


Fig. 80. — Circular conduit, partly full.

EXAMPLE II

Find the hydraulic radius of a circular pipe 10 ft. in diameter, and flowing 7.5 ft. deep.

The cross-section is shown in Fig. 80. Angle ABC is 30°; therefore, angle ACB is 120°, and the wetted perimeter is $\frac{2}{3} \times 10\ \pi = 20.94$ ft. The area

of triangle ABC is $2.5 \times 4.33 = 10.82$ sq. ft. The area of the remainder of the cross-section is two-thirds of the area of the circle, or $\frac{2}{3} \times 78.54 = 52.36$ sq. ft. Therefore, the total area is 63.18 sq. ft. and $R = 63.18 \div 20.94 = 3.02$ ft.[1]

Note that A being an area and P_w a length, R is a length, as its name indicates.

PROBLEMS

41-1. Find the hydraulic radius for a rectangular channel 20 ft. wide when flowing 10 ft. deep. *Ans.* 5 ft.

41-2. Solve the preceding problem when the depth is 5 ft. *Ans.* 3.33 ft.

41-3. Solve the preceding problem when the depth is 12 ft. *Ans.* 5.45 ft.

41-4. Find the ratio of hydraulic radius to wetted perimeter in the three preceding problems. Of all possible depths in a rectangular channel, which gives the maximum ratio? *Ans.* 0.125; 0.111; 0.124; half the width.

41-5. Solve Example I for a depth of 6 ft. *Ans.* 3.71 ft.

41-6. Solve Example I for a depth of 10 ft. *Ans.* 5.64 ft.

41-7. Find R for a circular conduit 6 ft. in diameter, flowing 4 ft. deep. Also when flowing 3 ft. deep.

41-8. In the pipe of Example II, find the depth that will give the maximum hydraulic radius. Solve either by calculus or by trial.

42. Chezy's Formula.—As noted in Art. 29, Chezy, in 1775, proposed a formula for the loss of head in open channels. In our notation it is

$$V = C\sqrt{R\,S} \tag{42.1}$$

where $S = \frac{h_f}{L}$. A circular pipe running full has a hydraulic radius of $\frac{D}{4}$ (prove). Making this substitution and solving simultaneously with equation (29.1), we see that the two equations are identical if

$$C^2 = \frac{8\,g}{f} \qquad \text{or} \qquad C = \frac{16.04}{\sqrt{f}} \text{ (in English units)} \tag{42.2}$$

[1] Values for the hydraulic radius at various depths of flow for both circular and "horseshoe" conduits are tabulated in Tables 12 and 12(b) of *Hydraulic and Excavation Tables*, Bureau of Reclamation, U. S. Dept. of the Interior (1934), obtainable from the Commissioner of the Bureau for $1.50. Values for other forms of cross-section are to be found in Swan and Horton's *Hydraulic Diagrams*, McGraw-Hill Book Co.

We should expect, then, a mode of variation of C with the size and roughness of the channel opposite to that of f in pipes. This proves to be the case, C being higher the smoother the channel walls and the larger the hydraulic radius. Some of the formulas which have been proposed for evaluating C are discussed in the next article.

EXAMPLE [2]

On May 10, 1912, the main channel of the Mississippi River above Carrollton, Louisiana had a measured discharge of 1,277,000 c.f.s. The cross-sectional area was 202,000 sq. ft., the wetted perimeter 2661 ft., and the slope 0.186 ft. per mile. Find Chezy's C.

$S = 0.186 \div 5280 = 0.0000352$. $R = 202{,}000 \div 2661 = 75.91$ ft. $R\,S = 0.00267$, $\sqrt{R\,S} = 0.0517$, $V = 1{,}277{,}000 \div 202{,}000 = 6.32$, $C = 6.32 \div 0.0517 = 122$.

A word may be said as to the units of C. Since V is in ft. per sec., S is an abstract number, and R is in ft., C will be in ft.$^{0.5}$ divided by sec. This makes C^2 in ft. per sec.^{2} as is required by the first part of (42.2) to make f an abstract number. The same channel will therefore have different values of C in different systems of units.

PROBLEMS

42–1. On April 2, 1915, the Ohio River at Cincinnati discharged 52,870 c.f.s., with a cross-sectional area of 16,800 sq. ft., a wetted perimeter of 1200 ft., and a slope of 0.355 ft. per mile. Find C. *Ans.* 103

42–2. On February 1, 1916, the Miami River at Dayton, Ohio, discharged 43,300 c.f.s., with a cross-sectional area of 7348 sq. ft., a wetted perimeter of 615 ft., and a slope of 1.938 ft. per mile. Find C. *Ans.* 89.0

42–3. On November 10, 1914, the Miami River cut-off channel below Dayton, discharged 458.2 c.f.s., with a cross-section of 82.0 sq. ft., a wetted perimeter of 50.6 ft., and a slope of 32.75 ft. per mile. Find C. *Ans.* 55.7

42–4. In the March, 1913, flood, Spring Creek, above Dayton, had a cross-sectional area of 1280 sq. ft., a wetted perimeter of 422 ft., and a slope of 4.6 ft. per 1000 ft. If the discharge was 5270 c.f.s., what was Chezy's C? *Ans.* 34.9

43. Kutter and Manning Formulas.—It is obvious that the Chezy formula is of little use in estimating flows unless some more definite information as to the value of C can be given. Be-

[2] The data for this example and the problems of this article are from *Calculation of Flow in Open Channels*, by Ivan E. Houk; Part IV of Technical Reports of the Miami Conservancy District (1918).

ginning about 1833 a number of studies of the variation of C with size, roughness, and other factors were made both by French and American engineers, but none of the formulas proposed before 1869 has proved useful. In that year the Swiss engineers, E. Ganguillet and W. R. Kutter (1818–1888), published the result of their study of practically all of the measurements of the flow of water in conduits, canals, and rivers made up to that time, and proposed a formula which, in English units and our notation, is

$$C = \frac{41.65 + \dfrac{0.00281}{S} + \dfrac{1.811}{n}}{1 + \left(41.65 + \dfrac{0.00281}{S}\right)\dfrac{n}{\sqrt{R}}} \tag{43.1}$$

Here n is a measure of the roughness, now called "Kutter's n." Numerical values are tabulated at the close of this article.

In spite of the cumbersomeness of this formula and the fact that it is not dimensionally homogeneous, it has been probably the most widely used formula for open channel flow, at least among American engineers. The research referred to in Note 2 reported that it fitted the data there studied better than any of the other twenty-odd formulas investigated.

However, there is very great doubt if C is a function of S at all except in small smooth channels. Even in the formula, it has little effect on the value of C unless S is very small. As a matter of fact, the slope term was introduced into the formula to account for the results of some float gagings of the Mississippi River by Humphreys and Abbot, which are now known to have been quite inaccurate. Therefore the formula could just as well be simplified to, say,

$$C = \frac{42 + \dfrac{1.811}{n}}{1 + \dfrac{42\, n}{\sqrt{R}}} \tag{43.2}$$

The advantage of the simplification is not great, because so many tables and charts of Kutter's formula have been published that it is seldom necessary to use the formula itself.

In 1890, Robert Manning [3] proposed a formula which was later modified by Parker [4] into

$$V = \frac{1.49}{n} R^{2/3} S^{1/2} \tag{43.3}$$

which is today called Manning's formula. It rests on the Chezy formula, and the empirical relationship

$$C = \frac{1.49}{n} R^{1/6} \tag{43.4}$$

The latter could have been more simply stated $C = K R^{1/6}$, but Parker was probably wise in putting it in the other form, as engineers were already familiar with values of n to be used in Kutter's formula for various roughnesses, and Parker's proposal was to use exactly the same value in equation (43.3).

In spite of the fact that the study referred to in Note 2 reported it unsatisfactory except for small channels, Manning's formula is coming more and more into use in American practice. The writer does not consider it the final word.

When the flow is known, but the depth unknown, the problem has to be solved by trial or by plotting a curve of Q against D. A sample is given in Problem 43–11.

EXAMPLE

Find n in Kutter's formula for the example of the previous article; also n by Manning's formula.

Since $S = 0.0000352$, $41.65 + \frac{0.00281}{S} = 121.48$. $R = 75.91$; therefore, $\sqrt{R} = 8.71$. Substituting in (43.1) $122 = \dfrac{121.48 + \dfrac{1.811}{n}}{1 + \dfrac{121.48\,n}{8.71}}$. Therefore, $122 + 1702\,n = 121.48 + \frac{1.811}{n}$ or $1702\,n^2 + 0.52\,n - 1.811 = 0$, and $n = \dfrac{-0.52 + \sqrt{0.27 + 12{,}329}}{3404} = 0.0325$ for Kutter's formula. For Man-

[3] "On the Flow of Water in Open Channels and Pipes," *Trans. Inst. of Civil Engineers of Ireland*, Vol. 20 (1891), p. 161, and Vol. 24 (1895), p. 179.

[4] *Control of Water*, by Philip à Morley Parker, Van Nostrand (1915), p. 472.

TABLE III.—RECOMMENDED VALUES OF n: [5]

Surface	Condition		Recommended Values
	Best	Bad	
Smooth brass and glass pipe	0.009	0.013	0.010
Neat cement surfaces	.010	.013	.011
Wood stave pipe	.010	.013	.011
Smooth lockbar and welded pipe	.010		.012
Planed plank flumes	.010	.014	.012
Unplaned plank flumes	0.011	0.015	0.013
Vitrified sewer pipe	.010	.017	.013
Semicircular metal flumes, smooth	.011	.015	.013
Cement mortar surfaces	.011	.015	.013
Coated cast-iron pipe	.011		.013
Glazed brickwork	0.011	0.015	0.013
Uncoated cast-iron pipe	.012	.015	.014
Commercial W.I. pipe, black	.012	.015	.014
Common clay drainage tile	.011	.017	.014
Commercial W.I. pipe, galvanized	.013	.017	.015
Brick in cement mortar	0.012	0.017	0.015
Concrete pipe	.012	.016	.015
Plank flumes with battens	.012		.015
Concrete-lined channels	.012	.018	.015
Dressed-ashlar surface	.013	.017	.015
Riveted and spiral steel pipe	0.013		0.016
Cement-rubble surface	.017	0.030	.0225
Earth, straight and uniform	.017	.025	.0225
Corrugated metal flumes	.0225	.030	.025
Winding sluggish channels	.0225	.030	.025
Dredged earth channels	0.025	0.033	0.0275
Dry-rubble surface	.025	.035	.030
Earth bottom, rubble sides	.028	.035	.030
Rock cuts, smooth and uniform	.025	.035	.033
Canals with rough beds	.025	.040	.035
Rock cuts, jagged and irregular	.035		.040
Natural Stream Channels			
(1) Clean, straight, full stage	0.025	0.033	0.030
(2) Same as (1), but some weeds and stones	.030	.040	.035
(3) Winding, some pools and shoals, clean	.033	.045	.040
(4) Same as (3), but some weeds and stones	.035	.050	.045
(5) Same as (4), but more stones	.045	.060	.050
(6) Sluggish river reaches, weedy	.050	.080	.070
(7) Very weedy reaches	.075	.150	.100

[5] Based on "Some Better Kutter's Formula Coefficients," by Robert E. Horton, *Eng. News,* February 24 and May 4, 1916.

ning's, $R^{1/6} = 8.71^{1/3} = 2.06$, and $n = \dfrac{1.49 \times 2.06}{122} = 0.0252$. This was a somewhat extreme case of large size and small slope. The two formulas generally give results that are more nearly equal.

PROBLEMS

43-1. For what range of values of S, does $41.65 + \dfrac{0.00281}{S}$ equal 42 within 1 per cent? *Ans.* All values of S greater than 0.0036

43-2. Compare equation (43.1) and (43.2) when $R = 1$ meter.

43-3. For the data of the example, what is n in equation (43.2)? *Ans.* 0.0198

43-4. The Humphreys and Abbot (1851) gaging nearest like the above example showed $R = 72.03$ and $S = 0.00002051$. Find V by means of Kutter's formula, Kutter's formula with slope omitted (43.2), and Manning's formula, using the n's found in the example and the preceding problem. *Ans.* 5.35 ft./sec.; 4.70 ft./sec.; 4.63 ft./sec.

NOTE. The measured mean velocity was 5.929 ft. per sec. In the light of this and other current meter measurements, it seems that either Humphreys and Abbot measured their velocity too high or their slope too small.

43-5. Find n in Kutter's formula and in Manning's formula for Prob. 42-1. *Ans.* 0.0243; 0.0226

43-6. Do the same for Prob. 42-2. *Ans.* 0.0253 in both.

43-7. Find n in Kutter's formula without slope (43.2) and in Manning's formula for Prob. 42-3. *Ans.* 0.0273; 0.0291

43-8. Do the same for Prob. 42-4. *Ans.* 0.0509; 0.0514

43-9. A trapezoidal irrigation canal accurately cut through earth has a bottom width of 24 ft., side slopes of 5 on 12, and a longitudinal bottom slope of 1 ft. per 1000 ft. Calculate the flow when the water is 5 ft. deep by Manning's formula. *Ans.* 885 c.f.s.

43-10. Find the flow in a trapezoidal concrete lined channel with a bottom width of 10 ft., side slopes of 1 to 1, a depth of 5 ft., and a longitudinal slope of 1 ft. in 2500 ft. *Ans.* 317 c.f.s.

43-11. How deep would the water be in Prob. 43-9 for a flow of 1200 c.f.s.?

44. A Suggested Formula.—In view of the very large number of formulas which have been proposed for calculating the flow in open channels it may be presumptuous to propose another. But the fact that none of those so far put forward seems to be entirely satisfactory leads the writer to suggest the following.

A circular pipe running half-full gives an open channel whose cross-sectional area is a semicircle and whose hydraulic radius is half the radius of the pipe. Both theory and experiment indicate that the same pipe flowing full will have virtually the same loss

per foot. Then (42.2) should apply. Also, if the walls are rough, and the channel reasonably large and the slope not too small, (30.1) will also apply. Combining the two

$$C = 32\left(\log_{10}\frac{r_0}{\epsilon} + 0.87\right) \tag{44.1}$$

Substituting $r_0 = 2\,R$ and simplifying

$$32 \log \frac{R}{\epsilon} + 37.5 \tag{44.2}$$

It might be assumed that these two formulas apply to all shapes of cross-section, but in order to make them fit observed measurements at least one other correction is found to be necessary. When the channel is wide with respect to its hydraulic radius, it is found that there is a tendency for C to be less than for narrower or deeper sections. This is taken care of (approximately, at least) by writing the formula

$$C = 32 \log \frac{R}{\epsilon} + 38 - \frac{B}{8\,R} \tag{44.3}$$

where B is the width of the free surface. For a square or circular conduit running half-full, $B = 4\,R$, and $38 - \dfrac{B}{8\,R} = 37.5$ as in (44.2). Equation (44.3) was derived from experiments in which B was more than $4\,R$ and should not be used where it is less. If such a case is encountered, use (44.2) for want of anything better.

It should be realized that this formula is only tentative,[6] and not based on the large amount of careful and expensive studies that would be necessary to verify it. It is the writer's opinion that the "true" formula for flow in rough channels lies in this direction. The question of flow in smooth channels is discussed in Art. 58. The range of values of ϵ are illustrated in the following

[6] Since the above was written, a very interesting publication has appeared—*Laws of Turbulent Flow in Open Channels* by Garbis H. Keulegan, Research Paper RP1151 of the National Bureau of Standards (Vol. 21, Dec., 1938). Keulegan's equation (60) when put in our notation, becomes $C = 32.6 \log\left(\frac{R}{\epsilon}\right) + 35.4$, which furnishes a fairly satisfactory check on the above.

example and problems. One reason that natural streams might not agree with (44.3) is that the energy loss is only partly due to turbulence resulting from the roughness of the sides and bottoms. It is also partly due to bends, to continual changes in cross-section equivalent to expansions and contractions, and often to breaks in the profile, so that the river consists of a series of pools and riffles.

EXAMPLE

Darcy and Bazin [7] in their run 27–9 on a semicircular test channel lined with pebbles varying from 3/8 to 7/8 in. in diameter held in place with cement, found a hydraulic radius of 0.968 ft., surface width of 4.0 ft., a slope of 1.5 per thousand, and a mean velocity of 3.73 ft. per sec. Find C, and ϵ in (44.3).

$$C = \frac{V}{\sqrt{RS}} = \frac{3.73}{\sqrt{0.968 \times 0.0015}} = 97.9$$

$\dfrac{B}{8R} = \dfrac{4.0}{8 \times 0.968} = 0.517$, $38 - \dfrac{B}{8R} = 37.5$, $\log \dfrac{R}{\epsilon} = \dfrac{97.9 - 37.5}{32} = 1.888$, $\dfrac{R}{\epsilon} = 77.27$, and $\epsilon = \dfrac{0.968}{77.27} = 0.0125$ ft. $= 0.15$ in. This is smaller than the pebbles, but since the pebbles were undoubtedly relatively smoother than Nikuradse's sand grains, and probably embedded to a larger proportion of their diameter, the result is not surprising.

PROBLEMS

44–1. Darcy and Bazin's run 27–2 had a mean velocity of 2.50 ft. per sec., a hydraulic radius of 0.546 ft., a surface width of 3.4 ft., and other data as in the example. Find C and ϵ. *Ans.* 87.4; 0.0148 ft.

44–2. Darcy and Bazin's run 12–1 was in a rectangular channel of boards 6.43 ft. wide, with wooden laths 1 × 3/8 in. nailed crosswise on the bottom and sides 3/8 in. apart. This channel gave a mean velocity of 1.65 ft. per sec. when flowing 0.33 ft. deep with a slope of 0.0015. Find C and ϵ, and compare with the depth of laths.

Ans. 77.9; 0.0140 ft.; the laths are too close together to have their full effect.

44–3. Darcy and Bazin's run 16–3 was on a channel similar to the above, but with the laths 2 in. apart. The width was 6.44 ft., R was 0.533 ft. and C was 59.0. Find ϵ and compare with depth of laths.

Ans. 0.106 ft.; with this spacing ϵ is more than the height of the laths. Streeter [8] found a similar result in pipes.

[7] *Recherches Hydrauliques*, Paris, 1865. Given in English units in Hering and Trautwine's translation of Ganquillet and Kutter's *Flow of Water*, John Wiley & Sons (1893), pp. 180–181.

[8] "Frictional Resistance in Artificially Roughened Pipes," by Dr. Victor L. Streeter; *Trans. A.S.C.E.*, Vol. 101 (1936), p. 692.

44–4. Taking ϵ for concrete lined channels as 0.005 ft., solve Prob. 43–10 by (44.3). *Ans.* 335 c.f.s.

44–5. Solve the preceding problem if $\epsilon = 0.01$ ft. *Ans.* 293 c.f.s.

44–6. What value of ϵ would in this case correspond to the $n = 0.015$ used in Prob. 43–10? *Ans.* 0.0080

45. Non-uniform Flow.—As has been mentioned above, natural channels have non-uniform cross-sections so that the average velocity is constantly varying to a greater or less extent. Even in a channel of uniform cross-section the same thing is often the case due to the fact that the liquid surface is not parallel to the bottom surface. For any given flow in a given channel there is one depth at which the velocity will be such that the slope of the

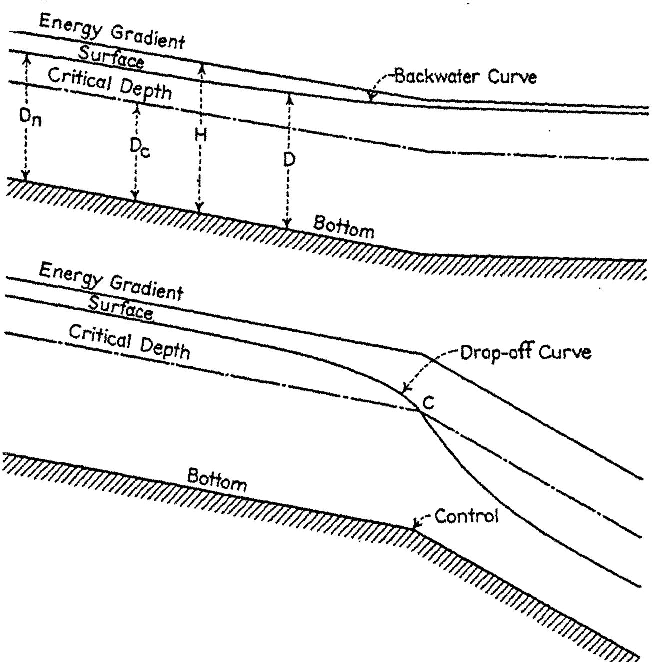


FIG. 81.—Backwater and drop-off curves.

energy gradient is the same as the bottom slope. The flow will then be uniform, and the depth is called the *normal* or *neutral* depth for that flow. This is the only depth at which this quantity of flow can be maintained throughout a long channel at this uniform slope. If at any point the bottom slope of the channel

increases, the liquid will accelerate and the surface slope will become steeper and the depth less. The profile of the surface in this case is called a *drop-off* curve. On the other hand, if the bottom slope decreases, the liquid will slow up, the surface slope will become less, and the depth increase. This gives what is called a *backwater* curve. These are illustrated in Fig. 81.

In this figure the *critical depth* is also indicated. As shown in Art. 27, for a rectangular channel with $\alpha = 1$, this is two-thirds the total specific energy. The critical depth for other cases is found as follows: The total specific energy, H, is the sum of the depth, D, and the velocity head, $\alpha \frac{V^2}{2g}$. But $V = \frac{Q}{A}$; therefore,

$$H = D + \frac{\alpha Q^2}{2 g A^2} \tag{45.1}$$

The critical depth for any given flow is the depth which makes this a minimum. This occurs when $\frac{dH}{dD} = 0$, or

$0 = 1 - \frac{2 \alpha Q^2}{2 g A^3} \times \frac{dA}{dD}$. But $\frac{dA}{dD} = B$, the width of the free surface; therefore, $1 = \frac{B \alpha Q^2}{g A^3}$, $\frac{A^3}{B} = \frac{\alpha Q^2}{g}$, and

$$\frac{A}{B} \qquad \alpha V^2 \tag{45.2}$$

Therefore, the *average* depth for critical flow is twice the velocity head, or two-thirds the energy head for all values of α and all shapes of cross-section. For open channels, α generally lies between 1.1 and 1.2, but it is sometimes as high as 2.0. Most discussions of the subject ignore α (that is, take it as one), but this is an approximation which is probably not usually warranted. It should be emphasized that the critical depth is a function of only three things, the amount of flow, the dimensions of the cross-section of the channel, and the velocity distribution; and that it is entirely independent of the slope and of the roughness, except in so far as these effect the velocity distribution.

For any given flow, dimensions, and roughness, there is one slope that will maintain the depth at the critical depth; that is,

the normal depth and the critical depth will coincide. This is called the *critical slope*. When the slope is less than the critical slope, as is usually the case, it is called a *mild* slope and the flow is called *tranquil* or *streaming;* when it is greater than the critical

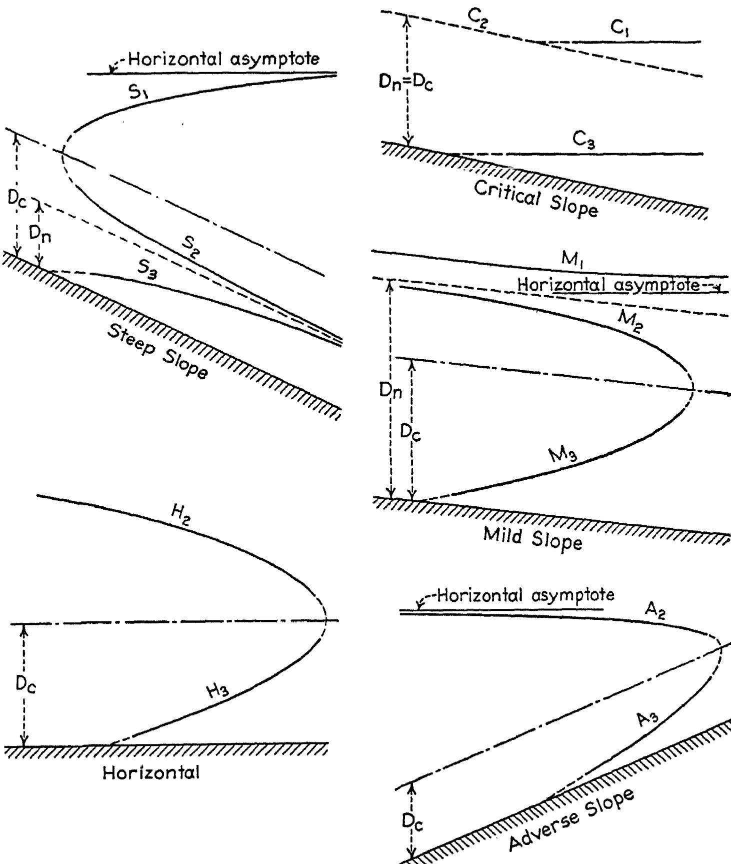


FIG. 82.—Types of surface curves in non-uniform flow.

slope, it is called a *steep* slope and the flow is called *rapid* or *shooting*. Figure 82 shows the various types of flow that may occur with these slopes and also with a horizontal bottom and with bottom sloping up in the direction of the flow (*adverse*). In general there are three types of flow: (1) in which the depth is greater than

normal and also greater than critical; (2) in which the depth is between normal and critical; and (3) in which it is less than either. In the figure, the various curves are drawn and labeled with the initial letter of the sort of slope, the subscript indicating the type just mentioned. There is no H_1 and A_1, because uniform flow cannot be maintained in a horizontal or adverse channel. Or, to state it differently, D_n is infinite for a horizontal channel and non-existent for an adverse one. The lower parts of the number (3) curves have been dashed because while mathematically they can be extended to intersect the floor, physically there must always be some depth if there is any flow. These curves are illustrated by the upper surface of the jet which issues from a partially opened sluice gate in a flume. Also, the portion of the curves near critical depth is shown dashed because the two can never join. The lower part of Fig. 81 shows an M_2 merging into an S_2, but S_1 could not change to S_2, etc. Uniform flow at critical depth is extremely unstable; standing waves tend to form on the surface, so C_2 is also shown dashed.[9]

The most satisfactory method of tracing a liquid surface through a channel of uniform cross-section is illustrated by Example I, following. The computation should always proceed upstream if the depth is greater than critical, or downstream if the depth is less than critical, as the effect of any error in the original starting level (or in the computation) will then tend to get less as one proceeds. In the case of natural streams, where the dimensions of the cross-section are known only at certain points, the "reaches" will have to terminate at these points. A first assumption as to the elevation will then have to be made, the area and hydraulic radius of the section found, and the slope computed. Ordinarily the elevation computed from this slope will not agree with the assumption, and the true level must be found by successive approximations. If, in computing the profile

[9] The classification here used is based on Dr. Rouse's extension of Dr. Bakhmeteff's classification with one change suggested by Professor C. J. Posey. Professor Woodward used a different letter for each of his twelve cases. See *Theory of the Hydraulic Jump and Backwater Curves*, by Sherman M. Woodward, Miami Conservancy Dist., Technical Reports, Part III (1917); *Hydraulics of Open Channels*, by Boris A. Bakhmeteff, McGraw-Hill Book Co. (1932); *Fluid Mechanics for Hydraulic Engineers*, by Hunter Rouse, McGraw-Hill Book Co. (1938), p. 293; "Backwater Curves in Theory and Practice," by C. J. Posey, Hydraulic Conference, Iowa City (1939).

of the water surface for tranquil flow in a stream of mild slope, it should appear that a point is lower than the immediately downstream point, change it to the same level. This occurs when the downstream point is in a large section of low velocity, and the upstream point is in a small section of high velocity. The assumption that all the velocity head is recovered would not be justified.

Similarly, where a narrow lined channel widens out into a wider unlined channel, computations may show that the slowing up of the water (due to greater roughness) may convert enough velocity head into depth so that the surface will actually rise. Water does sometimes flow up hill, but it would probably not do so in this case. Enough extra turbulence would probably be produced and α increased enough to produce a level surface. If it is desired to conserve the head, the transition should be made gradual enough so that Bernoulli's equation will give a level surface. See Example III.

EXAMPLE I

Let the lower part of Fig. 81 represent a rectangular channel 20 ft. wide carrying 800 c.f.s. Let the bottom slope upstream from C be 0.001 and $\epsilon = 0.005$ ft. Find the critical depth, the critical slope, the normal depth, and the elevations of the curve of the water surface from C upstream. Assume $\alpha = 1.29 = 0.04\,g$.

Since ϵ is given, the slope of the energy gradient will be computed by the method of Art. 44. If the material had been specified, most engineers would have used Kutter's or Manning's formula. If the proper n were used the results would have been nearly the same.

From (45.2), $D_c = \frac{0.04\,g}{g}\left(\frac{800}{20\,D_c}\right)^2$, $D_c^3 = 64$, and $D_c = 4.00$. The velocity at C will be $\frac{800}{4 \times 20} = 10$ ft. per sec. The hydraulic radius will be $\frac{80}{28} = 2.857$, and Chezy's C will be $32 \log \frac{2.857}{0.005} + 38 - \frac{20}{8 \times 2.857} = 88.2 + 37.1 = 125.3$. Then $10 = 125.3\sqrt{2.857\,S_c}$, $2.857\,S_c = 0.006369$, and $S_c = 0.00223$. It is obvious from the figure that downstream from C the bottom slope is greater than this, and the flow is shooting; while upstream from C it is less than this, and the flow is tranquil. The normal depth is determined by the equation for velocity, $\frac{800}{20\,D_n} = C\sqrt{\frac{20\,D_n}{2\,D_n + 20} \times 0.001}$. Since C depends on the depth, this must be solved by trial. As a first trial take $D_n = 6.0$ ft. Then $R = \frac{20 \times 6}{12 + 20} = 3.75$, $\frac{B}{8\,R} = \frac{20}{30} = 0.67$, and

$= 32 \log \frac{3.75}{.005} + 37.3 = 32 \log 750 + 37.3 = 92.0 + 37.3 = 129.3$. The left-side of the equation is $\frac{800}{120} = 6.67$ and the right side is $129.3\sqrt{0.00375} = 7.91$. As a second trial, let $D = 5.0$. Then $R = \frac{20 \times 5}{10 + 20} = 3.33$, $\frac{B}{8R} = \frac{20}{26.64} = 0.75$, and $C = 32 \log \frac{3.33}{.005} + 37.25 = 32 \times 2.823 + 37.25 = 127.6$, and

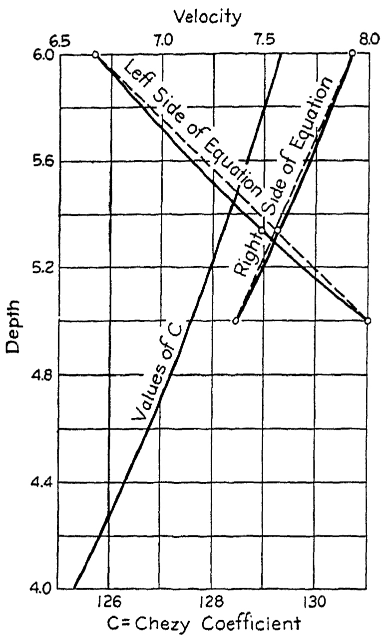


FIG. 83.

the right side of the equation becomes $127.6\sqrt{0.00333} = 7.36$. The left side is 8.00. Plotting these values of V against D in Fig. 83, the straight lines are found to intersect at $D = 5.34$ ft. This makes the left side of the equation $\frac{40}{5.34} = 7.49$. R is $\frac{20 \times 5.34}{10.68 + 20} = 3.48$ ft., $C = 32 \log \frac{3.48}{.005} + 38.0 - \frac{20}{8 \times 3.48} = 128.2$. and the right side of the equation becomes $128.2\sqrt{0.00348} = 7.56$. These points revise the straight lines into curves so that they intersect at $D = 5.30$ and $V = 7.54$. Therefore, the normal depth is 5.30 ft.

To trace the drop-off curve, assume a section L ft. upstream from C, where $D = 4.2$ ft. Then $R = \frac{84}{28.4} = 2.96$, and $V = \frac{800}{84} = 9.52$ ft. per sec. The values of C already computed are plotted against D in Fig. 83, a curve drawn through them, and from this C for $D = 4.2$ is read as 125.9. Then $S = \frac{9.52^2}{125.9^2 \times 2.96} = 0.00193$. The average slope of the energy gradient in this reach may be taken as $\frac{0.00223 + 0.00193}{2} = 0.00208$. The velocity head at C was $\frac{0.04\,g \times 10^2}{2\,g} = 2$ ft., therefore the total head there was $4.000 + 2.000 = 6.000$ ft. At the point where the depth is 4.2 ft., $V = 9.52$ ft. per sec. and $h_v = \frac{0.04\,g \times 90.63}{2\,g} = 1.812$ ft., and $H = 4.200 + 1.812 = 6.012$ ft. Then we have $0.00100\,L + 6.012 = 6.000 + 0.00208\,L$, and $L = \frac{0.012}{0.00108} = 11.1$ or say 11 ft.

Then investigate the point where $D = 4.40$ ft. Here $R = \frac{88}{28.8} = 3.05$, and $V = \frac{800}{88} = 9.09$ ft. per sec. From the curve in Fig. 83, $C = 126.3$ and $S = \frac{82.63}{15{,}952 \times 3.05} = 0.00170$. The average of this and the previous slope is 0.00184, h_v is $\frac{0.04\,g \times 82.63}{2\,g} = 1.653$ ft., and $H = 4.400 + 1.653 = 6.053$. $6.053 - 6.012 = 0.041$ ft., and $L = \frac{0.041}{0.00084} = 49$ ft. Similarly, the steps of the following table are computed.

D	R	V	C	S	Ave. S	H	L	ΣL
4.00	2.86	10.00	125.3	0.00223		6.000		
4.20	2.96	9.52	125.9	0.00193	0.00208	6.012	11	11
4.40	3.05	9.09	126.3	0.00170	0.00184	6.053	49	60
4.60	3.15	8.70	126.8	0.00149	0.00160	6.114	102	162
4.80	3.24	8.33	127.2	0.00132	0.00141	6.189	183	345
5.00	3.33	8.00	127.6	0.00118	0.00125	6.280	364	709
5.20	3.42	7.69	128.0	0.00106	0.00112	6.383	858	1567
5.30	3.46	7.54	128.2	0.00100	0.00103	6.437	1800	3367

Theoretically the point where $D = 5.30$ would be at an indefinite distance upstream, but practically it may be taken as about 3367, or, say, 3400 ft., upstream from C, as shown by the above computation.

EXAMPLE II

Draw a figure representing the trapezoidal channel of Prob. 43–10 and find the critical depth for a flow of 800 c.f.s. and $\alpha = 1.15$.

$B = 10 + 2\,D$ and $A = D\,(10 + D)$. Then $\frac{D^3(10 + D)^3}{10 + 2\,D} = \frac{1.15 \times 800^2}{32.16} = 22{,}890$. This is a sixth degree equation and can be solved by trial.

If $D = 5$, $\frac{125 \times 3375}{20}$ 21,090.

If $D = 6$, $\frac{216 \times 4096}{22} = 40{,}210$.

If $D = 5.1$, $\frac{132.6 \times 3443}{20.2} = 22{,}610$.

If $D = 5.12$, $\frac{134.2 \times 3457}{20.24} = 22{,}920$.

By interpolation, $D_c = 5.119$, or to three figures, 5.12 ft.

PROBLEMS

45–1. For what flow would a depth of 6 ft. be the critical depth in the channel of Example I if α is still 1.29, *Ans.* 1470 c.f.s.

45–2. Solve the preceding problem for a depth of 1 ft.

Ans. 100 c.f.s.

45–3. Find the critical depth for a flow of 320 c.f.s. in the above problems.

45–4. In the trapezoidal channel of Example II, find the flow for which a depth of 4 ft. would be critical. (α is still 1.15.) *Ans.* 521 c.f.s.

45–5. In the preceding problem, find the critical depth for a flow of 1000 c.f.s.

45–6. If in Example I, the bottom slope downstream from C is 0.005 find the normal depth.

45–7. In the preceding problem, trace the water surface downstream from C, finding the location of the points where the water is 3.80 ft. deep, 3.60 ft. deep, etc.

45–8. The outlets of the Beach City dam of the Muskingum Watershed Conservancy District are 18 ft. wide and have a longitudinal slope of 0.00416. If $n = 0.013$, what is the depth and flow for which this is the critical slope? Assume $\alpha = 1.2$. *Ans.* About 0.10 ft. and 3 c.f.s.

45–9. Using the same value of α as in the preceding problem, find the flow when the critical depth is 4 ft. *Ans.* 745 c.f.s.

45–10. With this flow the floor is "steep." Below the exit it is still steeper, so that the water surface will be of the S_2 type. Starting with a depth of 4.0 ft. at the upstream end, and using depths of 3.8 ft., 3.6 ft., etc., trace the water surface through the 198 ft. of tunnel length.

Note. These problems have been simplified from the actual Beach City dam by ignoring the fillet at the lower corners and the flare at the exit.

EXAMPLE III

A concrete lined channel with bottom width of 50 ft., side slopes of 1.5 to 1, and a bottom slope of 0.0004 has a normal depth of 10 ft. and $n = 0.015$. It is to widen into a natural channel, the transition being dry rubble with the same side and bottom slopes, and with $n = 0.030$. Design the transition so that the water surface will be level throughout the transition. Let α equal 1.15. Use Manning's formula.

In the lined channel $A = 650$ sq. ft., $P_w = 50 + 10\sqrt{13} = 86.06$ ft., $R = \frac{650}{86.06} = 7.55$ ft., $R^{2/3} = 3.849$, $S^{1/2} = 0.02$, $V \; \frac{1.49}{0.015} \times 3.849 \times 0.02$ $= 7.65$ ft. per sec., and $h_v = \frac{1.15 \times 58.52}{64.32} - 1.046$. At this point, n changes to 0.030, $S^{1/2} = 0.04$, $S = 0.00160$, and $Q = 650 \times 7.55 = 4908$ c.f.s. Now assume another section, L ft. downstream, where the bottom width is 60 ft. As a first approximation consider the depth to be 10 ft. Then $A = 750$ sq. ft., $V = \frac{4908}{750} = 6.54$ ft. per sec., and $h_v = \frac{1.15 \times 42.77}{64.32}$ $= 0.765$ ft. At this section, $P_w = 60 + 10\sqrt{13} = 96.06$ ft., $R = \frac{750}{96.06} = 7.81$ ft., $R^{2/3} \; 3.937$, $S^{1/2} \; \frac{6.54 \times 0.030}{3.937 \times 1.49} = 0.03345$, and $S = 0.00112$. Then the average S for

this reach is $\frac{0.00160 + 0.00112}{2} = 0.00136$, and $0.00040\,L + 10 + 1.046 = 0.00136\,L + 0.769 + 10$, $0.00096\,L = 0.281$, and $L = 293$ ft. $0.0004 \times 293 = 0.12$ ft. = additional depth at second point not yet accounted for. This increases A to 760.8 sq. ft., decreases the velocity to 6.45 ft. per sec., and h_v to 0.744 ft. P_w becomes $60 + 10.12\sqrt{13} = 96.49$ ft., $R = 7.89$ ft., $R^{2/3} = 3.963$, $S^{1/2} = \frac{6.45 \times 0.030}{3.963 \times 1.49} = 0.03277$, and $S = 0.00107$. Then the average slope for the reach is 0.00134, and $0.0004\,L + 1.046 = 0.00134\,L + 7.44$, $0.00094\,L = 0.302$, and $L = 321$ ft. $0.0004 \times 321 = 0.13$ ft. additional depth. This is only 0.01 more than before, so it seems unnecessary to make a third trial; but L might be increased to 322 ft. This is the distance from the beginning of the transition to the point where the bottom width would be 60 ft. Similarly, the distance to a width of 70 ft., 80 ft., etc., can be found, and the velocity decreased to a value which will not erode a natural channel, and the water surface kept level throughout the transition.

PROBLEMS

45-11. In Example III, find the distance from the beginning of the transition to the point where the bottom width is 70 ft.

45-12. Continue the preceding problem to find where the bottom width is 80 ft.

46. The Hydraulic Jump.—This topic may perhaps be best introduced by a numerical example.

EXAMPLE I

In a rectangular channel one ft. wide, 16 c.f.s. of water is flowing one ft. deep, with $\alpha = 1.005$. Could this same quantity flow at any other depth and still have the same total energy?

$V = 16.00$ ft. per sec.; the velocity head is $\frac{1.005 \times 256}{2 \times 32.16} = 4.00$ ft.; and the total energy head is $1 + 4 = 5$ ft. Then if the depth is changed to D, the area is D, the velocity is $\frac{16}{D}$, the velocity head $\frac{4}{D^2}$, and $5 = D + \frac{4}{D^2}$ or $D^3 - 5\,D^2 + 4 = 0$. Since this is a cubic equation it has three roots, of which we know one to be 1.00. Dividing by $D - 1$ we get $D^2 - 4\,D - 4 = 0$, which has two roots, 4.828 and -0.828. Obviously the negative root has no physical meaning. But the 4.828 ft. is an *alternate depth* of flow for the same quantity and same energy. It will be found that by properly adjusting the slope of the flume and the tailwater level, either depth of flow could be maintained by the same inflow. One alternate depth is always less than D_c and the other more. (In the exceptional case where the depth of flow equals D_c, there is no alternate depth.) A mass of numerical illustrations is condensed into Fig. 84.

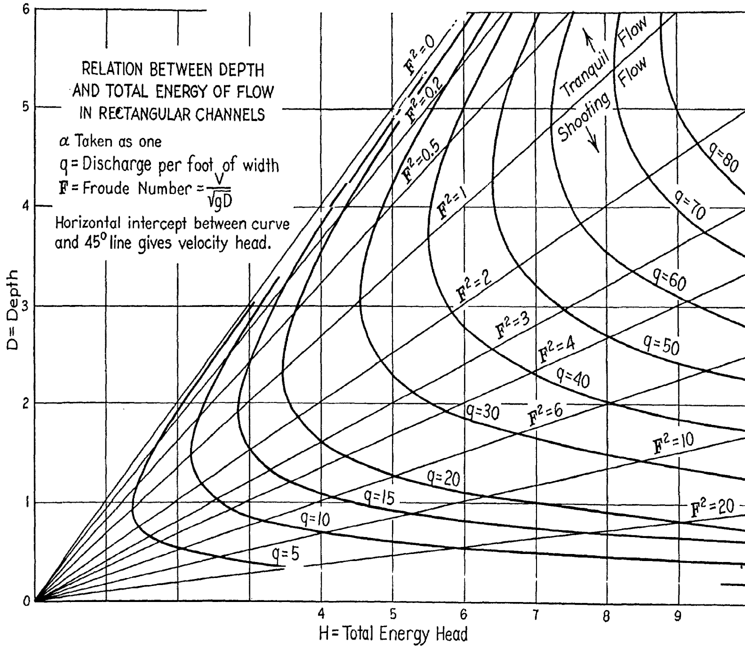


FIG. 84.—Specific energy curves for rectangular channels.

In some cases there is an endeavor on the part of liquid flowing at less than critical depth spontaneously to change to the alternate depth. This is never successful, because the process of change causes a loss of energy and H is thereby changed. But, neverthe-

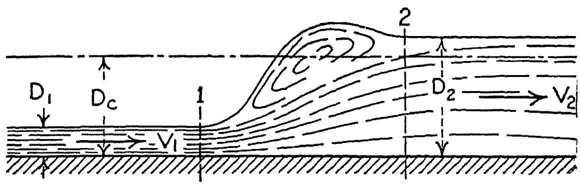


FIG. 85.—Diagram of hydraulic jump.

less, a change from a lower stage to a higher stage does occur in nature, and is called the *hydraulic jump*. The most common instance is below dams, either on the aprons of Ogee spillways or in the stilling basins of outlet works. The phenomenon is shown diagrammatically in Fig. 85, and photographically in Fig. 86.

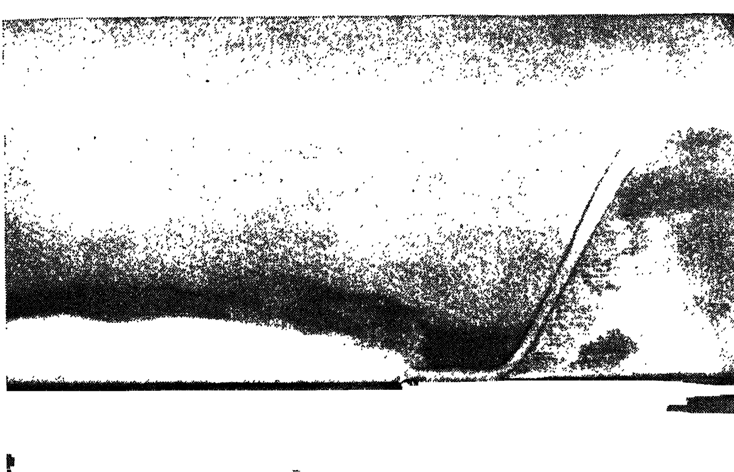


FIG. 86.—Hydraulic jump in demonstration flume. The model spillway section at the right is only 4.4 inches high.

A backward "roller" forms on the face of the jump and a great deal of air is temporarily mixed with the liquid, forming a foam which has a specific weight less than that of the liquid. The creation of this turbulence absorbs quite a little energy, which

is usually desired. The main reason why a jump is desired in a stilling basin is to reduce the velocity V_2 enough so that it will not damage the stream channel. This is done principally by changing from shooting to tranquil flow and is only secondarily due to energy loss.

A hydraulic jump will always form on a mild slope which continues far enough if the initial depth is six-tenths of critical or less. That is, the surface suddenly changes from M_1 to M_2 or M_3, or more commonly to normal depth. There can also be a change from S_3 or S_2 to S_1, from H_3 to H_2, or from C_3 to C_1. A rise in the tailwater will often "drown out" the jump, and this is almost certain to occur on an adverse slope unless the slope is very flat and changes to positive soon after the jump occurs.

Returning now to Fig. 85, the mass of liquid between sections 1 and 2 is held in equilibrium by the excess of the impulse of the liquid entering at 1 over that of the liquid leaving at 2. In other words, the momentum of the liquid leaving per second at 2 is less than the momentum of the liquid entering per second at 1 by the difference in the hydrostatic forces on the sections 2 and 1. This is expressed by the equation

$$\frac{w_1 Q\,\beta_1 V_1}{g} - \frac{w_2 Q\,\beta_2 V_2}{g} = A_2 w_2 \bar{y}_2 - A_1 w_1 \bar{y}_1 \tag{46.1}$$

where $\bar{y}_1$ and $\bar{y}_2$ are the distances that the centers of gravity of A_1 and A_2 are below the respective liquid surfaces. As noted above, w_2 will be less than w_1, but since the actual value is not known, w_2 is generally taken as equal to w_1 and thus cancelled out. It is also usual to take $\beta_1 = \beta_2 = 1$. Then for a rectangular channel, $\bar{y} = \frac{D}{2}$, and $\frac{Q}{g}(V_1 - V_2) = \frac{B}{2}(D_2^2 - D_1^2)$. But from the law of continuity (if $w_1 = w_2$),

$$V_2 = \frac{V_1 D_1}{D_2}, \qquad \text{and} \qquad \frac{Q\,V_1}{g}(D_2 - D_1) = \frac{B\,D_2}{2}(D_2^2 - D_1^2),$$

$$\frac{Q\,V_1}{g} = \frac{B\,D_2}{2}(D_2 + D_1), \quad \text{and} \quad D_2^2 + D_1 D_2 = \frac{2\,Q\,V_1}{B\,g} = \frac{2\,D_1 V_1^2}{g}$$

Solving this,

$$D_2 = \frac{-D_1 \pm \sqrt{D_1^2 + \frac{8\,D_1 V_1^2}{g}}}{2} \quad \text{and} \quad \frac{D_2}{D_1} = \frac{1}{2}\left(\sqrt{1 + \frac{8\,V_1^2}{g\,D_1}} - 1\right).$$

As stated in Appendix B, $\frac{V^2}{g\,D}$ is $\mathbf{F}^2$, where $\mathbf{F}$ is called the Froude number, therefore,

$$D_2 = \frac{D_1}{2}\left(\sqrt{1 + 8\,\mathbf{F}_1^2} - 1\right) \tag{46.2}$$

This equation is true only when $\mathbf{F}_1$ is more than one; that is, for shooting flow. When $\mathbf{F}_1^2$ is 3, it gives $D_2 = 2\,D_1$. This is about the lowest $\mathbf{F}_1$ for which a true jump will form.

EXAMPLE II

If a jump occurs in Example I, what will be the depth after the jump?

$$\mathbf{F}_1^2 = \frac{V_1^2}{g\,D_1} = \frac{256}{32.16 \times 1} \quad 7.96 \qquad D_2 = \frac{D_1}{2}\left(\sqrt{1 + 63.68} - 1\right) = 3.521\,D_1.$$

Then since D_1 was 1.00, D_2 will be 3.52 ft. Note that this is less than the 4.828 ft. found in Example I.

The problem as to when and where a jump will form will not be solved in this text. A complete discussion is given in the first three references of Note 9. In general the occurrence and position depends on the tail water. Whenever the tail water depth is D_2 a jump will form if the initial depth is D_1. In Prob. 46–6 a formula is derived for D_1 in terms of D_2. If the downstream flow is at normal depth, D_2 can be computed if the quantity, slope, and roughness are known. Then a surface curve for the initial flow is computed by the method of the preceding article, and the jump will occur wherever the depth becomes equal to D_1. If the flow downstream is not uniform but is governed by some control [10] downstream, a surface curve must be computed from the control upstream. Practical methods are described in the references given above.[11]

PROBLEMS

46–1. When the depth in Example I is changed to 4.828 ft., check to show that the total head remains unchanged.

46–2. What is the critical depth in Example I? *Ans.* 2.00 ft.

[10] The technical meaning of this term is discussed in the latter part of the next article.

[11] Equations for jumps in channels which are not rectangular are given in "The Hydraulic Jump in Standard Conduits" by J. C. Stevens, *Civil Engineering*, Vol. 3 (1933), pp. 565–567, and discussion by G. H. Hickox, *Civil Engineering*, Vol. 4 (1934), p. 270.

46–3. What is the efficiency of the jump in Example II; that is, the ratio of total energy head after the jump to total energy head before the jump? [12]

Ans. 76.9% assuming same α.

46–4. A 5 ft. sluice gate at the upper end of a flume 5 ft. wide is raised 1 ft. and 75 c.f.s. flows out. A jet forms which is at first only about 0.60 ft. deep. But the slope of the flume is not enough to maintain this velocity and the jet retards and becomes deeper. When it attains a depth of one foot, a jump forms. What is the depth after the jump? *Ans.* 3.27 ft.

46–5. From Fig. 84, find the total original head for the preceding problem, the critical depth, and the alternate depth corresponding to one foot; then compute the efficiency of the jump.

Ans. 4.5 ft.; 1.92 ft.; 4.31 ft.; 80.0%

46–6. Show that $D_1 = \frac{D_2}{2}(\sqrt{1 + 8\,\mathbf{F}_2^2} - 1)$.

46–7. Test the following approximate formula [13] for a rectangular channel, $J = \frac{D_2}{D_1} = 1.4\,\mathbf{F}_1 - 0.4$, by finding the per cent error for $\mathbf{F}_1 = \sqrt{3}$, 4, and 7.

Ans. 1.24%; 0.41%; − 0.13%

46–8. By writing Bernoulli's equation (with loss) between points 1 and 2 in Fig. 85 and putting $\frac{D_2}{D_1} = J$ show that the head lost in the jump is h_j

$\frac{D_1}{4}\left[J^2 - \frac{1}{J} - 3(J - 1)\right]$ and that the efficiency is

$$1 - \frac{\left[J^2 - \frac{1}{J} - 3(J - 1)\right]}{4 + 2\,\mathbf{F}_1^2}.$$

46–9. By the result of the previous problem, check the efficiencies in Probs. 46–3 and 46–5. The check will not be perfect because these formulas assume $\alpha = 1$.

46–10. Show that if $\frac{D_1 + D_2}{2}$ is written as D_a, $D_1 D_2 D_a = D_c^3$.

47. Flow in Streams.

—Information on the flow of water in streams is required for the proper design of navigation works, hydroelectric power plants, flood control works, and water supply

[12] It should not be supposed that all of the energy lost is converted into heat in the jump. Much of it goes into kinetic energy of rotation of eddies and excess kinetic energy of translation due to the pulsations in the three velocity components and to non-uniform distribution of velocity. However, the larger part of this energy will be converted into heat within a comparatively short distance downstream.

[13] This formula was derived by the author following a suggestion by Mr. Karl R. Kennison in *Civil Engineering*, Vol. 1 (1931), p. 543. Mr. Kennison was the pioneer American writer on the hydraulic jump, his first paper on the subject having been published in Vol. 80 of *Trans. A.S.C.E.* (1916). The Frenchman, Belanger, was the first to work out the theory of the hydraulic jump, his work having been published in 1838.

of all sorts, including irrigation. In the United States the collection of this information is in the hands of the Water Resources

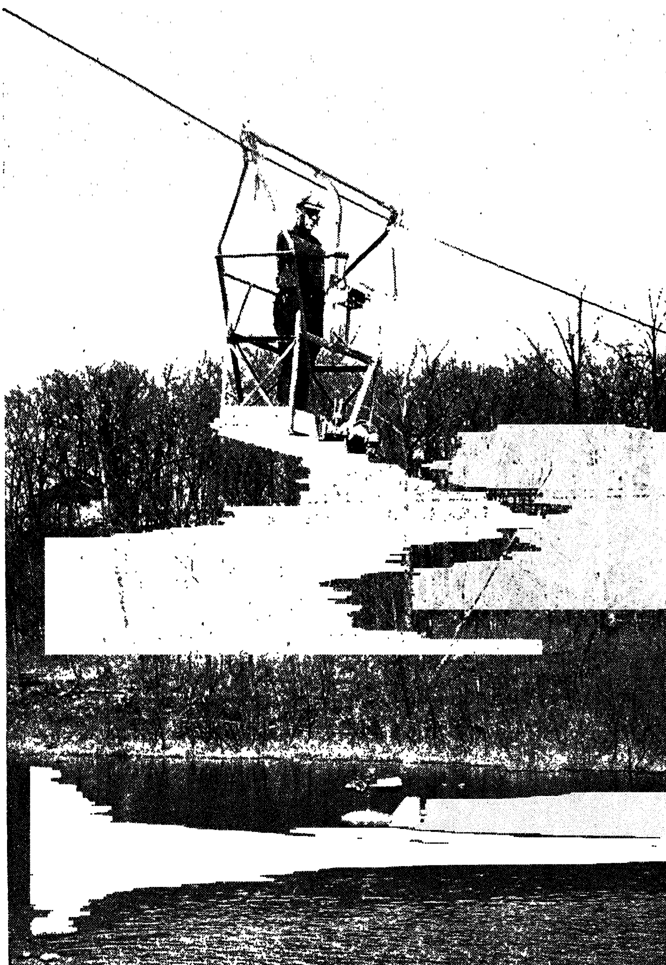


FIG. 87.—Gaging from a cable car. (Courtesy U. S. Geological Survey.)

branch of the Geological Survey.[14] In cooperation with the various states, the Survey assembles and publishes each year

[14] For a brief summary of this work see "Geological Survey Studies Surface Waters," by C. G. Paulsen, *Civil Engineering*, Vol. 8 (April, 1938), pp. 247–250.

the daily flow at some 3200 river-measurement stations. See Fig. 87. The method is described in Appendix D.

It would not be practicable to make a gaging at each station each day in the year. Instead, gagings are made only a few times a year, and a curve such as Fig. 88 of discharge against gage

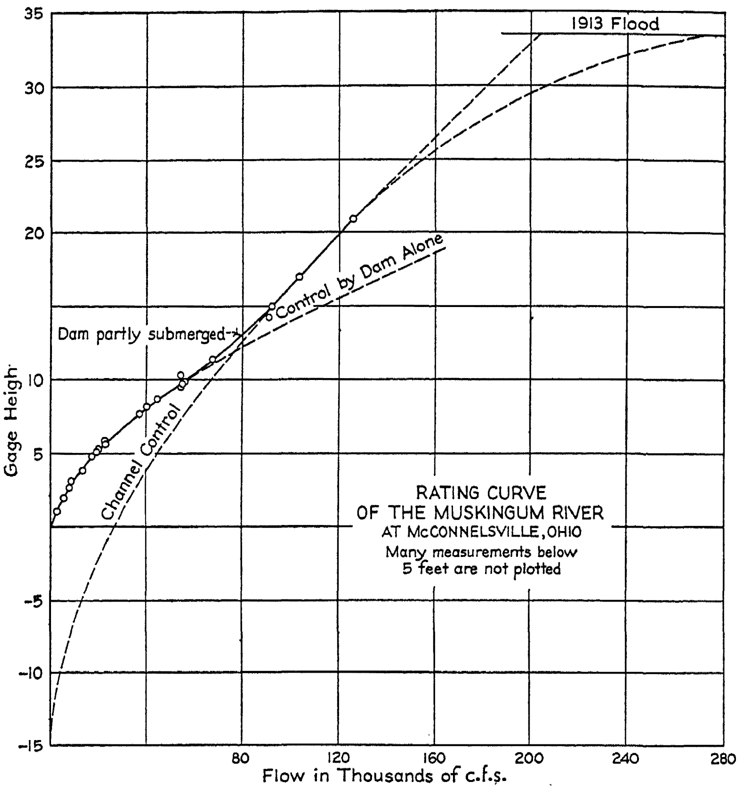


FIG. 88.—Rating curve illustrating changing control.

height plotted. This is called a *rating curve*. Then a gage reader reads the gage once or twice a day, or an instrument is arranged to make a continuous automatic record of the gage height (Fig. 89), and from it and the rating curve the discharge is found. However, it is necessary to take gagings at intervals, because the rating curves are often found to change with time. This brings up the important matter of *control*. This may be a place where the velocity gets up to critical, as at C in Fig. 81, or simply

a constriction in the channel, where the velocity is higher than it is immediately upstream (without getting to critical velocity). Downstream from any gaging station there is generally a riffle which acts as control for low stages. As long as this remains the same, the rating curve remains unchanged. At higher water,

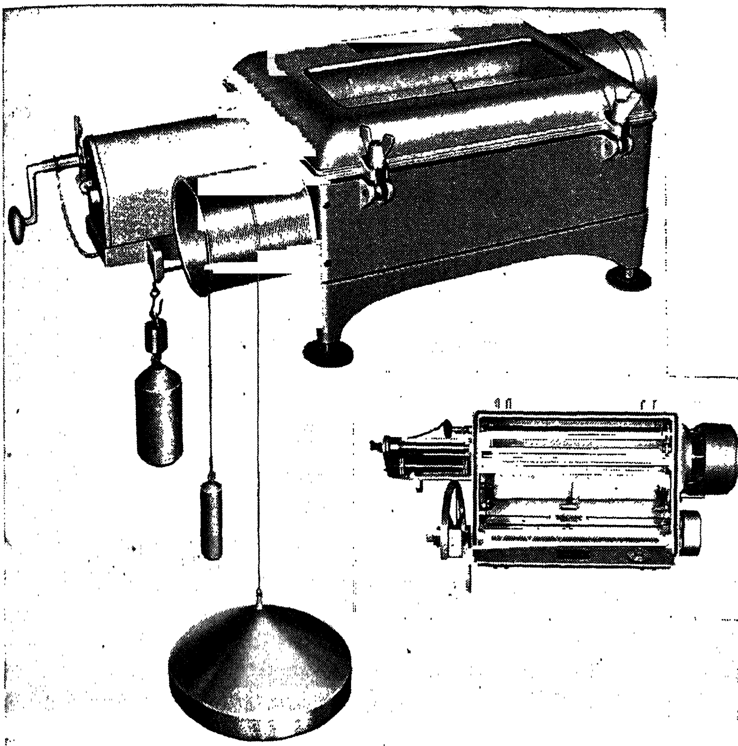


Fig. 89.—Water stage recorder. The inset shows the cover removed, and in place of the drum a sprocket wheel for use with a flat tape. (Courtesy of Julien P. Friez & Sons.)

some point still further downstream may act as control, or there may be no particular point, but simply "channel control." In low water in summer the growth of weeds may change the control. Figure 88 shows a somewhat special case. Here a navigation dam about 15 feet high forms the control at ordinary times. The zero of the gage in this case is the crest of the dam. But as the dam is old and leaks, there is some flow even when the gage

height is zero. Except for this, the discharge varies nearly as the three-halves power of the gage height up to about 10 feet. Here the tail water below the dam becomes high enough above the crest so that submergence begins to affect the flow. This is shown by the reversal of curvature in the rating curve. When the gage height reaches 17 feet (as it does about once a decade), the dam is completely "drowned out" and the control is simply the channel below. In Fig. 88 the curve from 17 feet to 21 feet has been extended backward (dashed) toward a point 15 feet below zero to indicate a guess as to what the rating curve would be if the dam were removed. It has also been extended in the other direction to the greatest known height (that of the March, 1913 flood). This has been done in two ways: one from a logarithmic plotting of Q and $(H + 15)$, indicating 204,000 c.f.s. for the 1913 flood; and the other to the United States Engineers' estimate of 272,000 c.f.s. This estimate was made before the two highest measurements shown on the curve were made, and now seems to be too high.

If possible, a gaging station should not be located where it is affected by backwater from other streams. If this cannot be avoided, a "slope station" is used; that is, two gages far enough apart to determine the slope are installed, and rating curves are developed for a number of different slopes, generally on the assumption that the discharge is proportional to the square root of the slope.

Even where there is no backwater, the slope is generally different on "rising stage" than on "falling stage," and the discharge for a given gage height depends on the rate of rise or fall.[15] An example and problems covering this article are given in Appendix D.

[15] For further information on this subject, see the latter part of the writer's discussion of "Flood Routing" in *Proc. A.S.C.E.* (Sept., 1938), pp. 1473–1474.

CHAPTER VI

PIPE FLOW—II

48. Viscosity.—We can no longer delay coming to grips with the distinctive property of fluids, viscosity. Figure 90 represents a rectangular portion of a fluid, $BCDE$, which is flowing horizontally toward the right, the upper portion flowing faster than the lower, so that the velocity at E is v feet per second faster than the velocity at B. Then at time dt later the fluid which was within the space $BCDE$ will be in the space $BCD'E'$, where $EE' = DD' = v\,dt$.

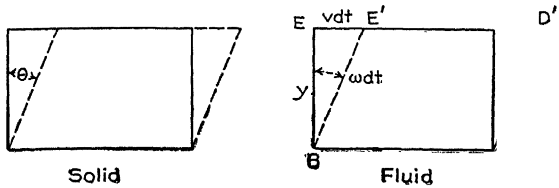


FIG. 90.—Analogy between viscosity and elasticity.

Now this will occur only when there is an external couple acting on the block of fluid, consisting of a force τA to the right along ED and τA to the left along BC, A being the area $BC = ED$ times the dimension perpendicular to the paper, and τ being the unit shearing stress (which in solids is generally represented by S_s). In most actual cases the shearing stress is not quite the same in the two layers, but that is a complication which need not concern us here.

If the height $BE = CD$ is represented by y, we can define ω as the ratio of v to y. It is the angular velocity with which the line connecting two points in the fluid which are momentarily abreast of each other is revolving due to the unequal velocities.

Sir Isaac Newton (English, 1642–1727) in his *Principia*, written about 1686, stated that the shearing stress and the angular velocity are proportional, and accurate measurements have since shown him to have been correct. That is, for uniform shear

(48.1) $$\tau = \mu\,\omega = \frac{\mu\, v}{y}$$

where μ (mu) is called the *coefficient of viscosity*, or the *absolute viscosity*.

During the past century a vast number of measurements of viscosity have been made, mostly by physical chemists. They have found that for almost all liquids the coefficient of viscosity decreases with a rise of temperature, and that, on the contrary, the viscosity of gases increases with a rise in temperature. Changes of pressure have little if any effect on the viscosity of most fluids. Two exceptions are lubricating oils, where high pressure increases the viscosity, and steam, where increasing the pressure from atmospheric to ten atmospheres, increases the viscosity about 20 per cent.

The likenesses and differences between solids and fluids may be noted at this point. In a solid, a given shearing stress (not exceeding the elastic limit) causes a definite deformation which takes place almost instantly, and after this there is no further change. But a fluid under a shearing stress continues to deform at a rate proportional to the stress as long as the stress continues. The coefficient of elasticity in shear for a solid, E_s, is the ratio of the unit shearing stress to the unit shearing deformation. The coefficiency of viscosity for a fluid is the ratio of the unit shearing stress to the *time rate* of unit shearing deformation.

Equation (48.1) may be written in the form

(48.2) $$\mu = \frac{\tau\, y}{v}$$

Equations (48.1) and (48.2) apply only when v varies directly as y. When this is not the case we put our planes an infinitesimal distance apart, and obtain the real definition of viscosity

(48.3) $$\mu = \tau\,\frac{dy}{dv}$$

Since τ is force divided by area $[F\,L^{-2}]$;[1] y is length $[L]$; and v is length divided by time $[L\,T^{-1}]$; μ has the dimensions of

[1] The square brackets here have the meaning "the dimensions of." Thus $[\mu] = [M\,L^{-1}T^{-1}]$ means "the dimensions of viscosity are mass divided by length and time." A further discussion is given in Appendix B.

$[T\,F\,L^{-2}]$. Therefore, for English absolute units, viscosity will be in seconds times poundals per square foot. But it is more usual to consider mass, length, and time as the fundamental units. Then $[F] = [M\,L\,T^{-2}]$ and $[\mu] = [M\,L^{-1}T^{-1}]$. Viscosity in the English absolute units may also be thought of as in pounds (of mass) per foot-second; and in the British engineers' units it will be in slugs per foot-second. This latter unit is g times as large as the former, so that numerical values of μ will be only about one thirty-second as large in the British engineers' units as in the English absolute.

In the c.g.s. system, μ can be thought of as in dyne-seconds per square centimeter, or as in grams (of mass) per centimeter-second. From either standpoint, the numerical value in any given case is the same, and the unit is called the *poise*, in honor of Poiseuille. For most fluids the viscosity is very much less than one poise, so that the common unit used by physical chemists is the *centipoise*, which is one-hundredth of a poise. Fortunately, the viscosity of pure water at a temperature of 68.6° F. is one centipoise, so that the viscosity in centipoises of any fluid is also its *relative* viscosity (relative to water at 68.6° F.).

There are many things in fluid mechanics which depend on the ratio of the viscosity to the density. This ratio has been given a name, *kinematic* viscosity and is represented by the letter ν (nu).

$$\nu = \frac{\mu}{\rho} \tag{48.4}$$

Density is mass per unit volume; therefore, $[\nu] = [M\,L^{-1}T^{-1}] \div [M\,L^{-3}] = [L^2T^{-1}]$, and kinematic viscosity is of the nature of an area divided by time. In the c.g.s. system it is in square centimeters per second (sometimes called a *stoke*, in honor of Sir G. G. Stokes who was the first to use it); and in all the English systems it is in square feet per second. Since there are almost exactly 929 square centimeters in a square foot, a kinematic viscosity of 1 square foot per second is equal to 929 stokes.

Table IV gives the absolute viscosity and kinematic viscosity of some common fluids in c.g.s. units. They are all for one atmosphere pressure and for 20° C. except steam, which is for 100° C. It must be realized that some of the substances (like crude oil) are variable, and the values given are only average or typical.

The viscosity of water at various temperatures and an alignment chart for finding the viscosity of various liquids at different temperatures are given in Appendix A.

TABLE IV.—VALUES OF VISCOSITY OF VARIOUS FLUIDS

FLUID	ABSOLUTE VISCOSITY IN POISES	KINEMATIC VISCOSITY IN STOKES
Hydrogen	0.00009	1.05
Steam	0.00014	0.23
Air	0.00018	0.150
Carbon bisulfide	0.00367	0.00280
Water	0.0101	0.0101
Mercury	0.0157	0.0011
California crude oil (light)	0.37	0.39
Linseed oil	0.38	0.41
Glycerine	8.7	6.9
Castor oil	9.86	10.27
California crude oil (heavy)	21	22

EXAMPLE

Find the number by which viscosity in centipoises must be multiplied to give viscosity in slugs per ft.-sec.; in lb. (of mass) per ft.-sec.; in sec.-lb. (of force) per sq. ft.; and in sec.-poundals per sq. ft.

Let x = the viscosity in centipoises. Then 0.01 x is the viscosity in poises; that is, gm. per cm.-sec. Since 1 gm. = 0.0022046 lb., the viscosity in lb. per cm.-sec. will be 0.000022046 x; and since 1 ft. = 30.48 cm., the viscosity in lb. per ft.-sec. will be $30.48 \times 0.000022046\, x = 0.0006720\, x$. Taking the number of lb. in a slug as 32.174, the viscosity in slugs per ft.-sec. will be $0.0006720 \div 32.174 = 0.0000209\, x$.

Or, starting over again by taking 0.01 x as the viscosity in sec.-dynes per sq. cm., the viscosity in sec.-dynes per sq. ft. will be 9.29 x; and since 1 dyne $= \dfrac{0.0022046}{980.66} = 0.000002248$ lb. (of force), the viscosity in sec.-lb. per sq. ft. will be $9.29 \times 0.000002248\, x = 0.0000209\, x$. Multiplying by 32.174 we get 0.000672 x sec.-poundals per sq. ft. To recapitulate—to reduce viscosity in centipoises to slugs per ft.-sec., or to sec.-lb. per sq. ft., multiply by 0.0000209. To reduce centipoises to lb. (of mass) per ft.-sec., or to sec.-poundals per sq. ft., multiply by 0.000672.

PROBLEMS

48-1. How many centistokes are there in a sq. ft. per sec.?

Ans. 92,900

48-2. Find the viscosity of water at 68° F. in slugs per ft.-sec. Do the same for mercury. *Ans.* 0.0000211; 0.0000328

48-3. Convert the values of kinematic viscosity given in Table IV for water, castor oil, and glycerine to English units.

Ans. 0.0000109 sq. ft./sec.; 0.0111 sq. ft./sec.; 0.00743 sq. ft./sec.

49. Non-turbulent Flow.—The flow in pipes considered in Chapter IV, where the resistance varies more or less as the square of the velocity, is the sort ordinarily occurring in engineering work. It is called *turbulent* flow. But as early as 1840, Gotthilf Hagen (German, 1797–1884) and Jean Louis Poiseuille (French, 1799–1869) had discovered independently that when water flows in

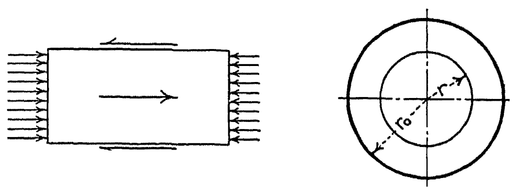


FIG. 91.—Part of liquid flowing in a pipe considered as free body.

very small tubes, the resistance varies as the first power of the velocity. Such flow is called *non-turbulent, laminar, viscous,* or *streamline* flow, because the liquid moves in parallel stream lines. In 1856 Wiedemann showed mathematically that if the flow takes place in this manner, the resistance must vary as the first power of the velocity. The proof follows.

In Fig. 91 we have a pipe of radius r_0 in which a liquid is flowing non-turbulently from left to right. At distance r from the center the velocity is v, and the shearing stress is τ. Take a cylinder of the liquid of radius r and length L as the free body. For steady flow this portion of the liquid will be in equilibrium, and if p_1 and p_2 represent the unit pressure at the upstream and downstream ends, respectively, $(p_1 - p_2)\, \pi r^2 = 2\, \pi\, r\, L\, \tau$, and

$$\frac{(p_1 - p_2)\, r}{2\, L} \tag{49.1}$$

That is, the shear is zero at the center of the pipe, and varies directly as the distance from the center, having its maximum value, τ_0, at the pipe wall where $r = r_0$. From (48.3), $\tau = \mu \dfrac{dv}{dy}$. But $y = r_0 - r$, therefore $dy = -\, dr$. Substituting these in

(49.1), $\frac{(p_1 - p_2)\,r}{2\,L} = -\,\mu\,\frac{dv}{dr}\cdot$ This shows that at the center of the pipe where $r = 0$, the velocity must be a maximum. We will call this velocity v_m. At the pipe wall the velocity will be zero.[2] Then integrating, $\frac{(p_1 - p_2)}{2\,L}\int r\,dr = -\,\mu\int dv$, and $\frac{(p_1 - p_2)\,r^2}{4\,L} = -\,\mu\,v + K.$ When $r = 0$, $v = v_m$, therefore $K = \mu\,v_m$, and

$$(49.2) \qquad v = v_m - \frac{(p_1 - p_2)\,r^2}{4\,\mu\,L}$$

When $r = r_0$, $v = 0$; therefore,

$$(49.3) \qquad v_m \quad \frac{(p_1 - p_2)\,r_0^2}{4\,\mu\,L}$$

Substituting this in (49.2),

$$(49.4) \qquad v = \frac{(p_1 - p_2)\,(r_0^2 - r^2)}{4\,\mu\,L}$$

This is the equation of a parabola; therefore, the volume of liquid flowing past any section per second is the volume of a paraboloid of revolution with base $\pi\,r_0^2$ and altitude v_m. The volume of such a paraboloid is one-half its base times its altitude; therefore,

$$(49.5) \qquad V = \frac{v_m}{2} = \frac{(p_1 - p_2)\,r_0^2}{8\,\mu\,L}$$

Then

$$(49.6) \qquad h_f = \frac{p_1 - p_2}{w} = \frac{p_1 - p_2}{g\,\rho} = \frac{8\,\mu\,L\,V}{g\,\rho\,r_0^2} = \frac{32\,\nu\,L\,V}{g\,D^2}$$

This shows what we set out to prove; namely, that the resistance

[2] This was first pointed out by Stokes (*Math. and Phy. Papers*, Cambridge University Press (1880–1905), p. 539). The reason there appears to be velocity at the walls is that the rate of increase of velocity there is so large that at a few thousandths of an inch from the wall the velocity may be several ft. per sec.

varies as the first power of the velocity. It also gives us a quantitative formula by which to forecast the flow for a given pressure drop, or the drop for a given flow, without any coefficient to be determined experimentally. This is what we should like to be able to do in every problem. Unfortunately, however, we have been successful in only a few cases of which this is one. And it is a real success, because experiments check the formula as closely as we can measure. Equation (49.6) is generally called Poiseuille's equation. The Hagen-Poiseuille equation would be a better name.

It might be added that actually the acceleration of gravity has no effect on the resistance to flow in pipes. This factor appears in (49.6) as it did in (29.1) only because we are measuring the resistance in feet of the liquid flowing. If we measured the resistance as a pressure drop, say in pounds per square inch, then g would not come in. μ would then be involved instead of ν.

EXAMPLE

Saph and Schoder's run No. 292 on Pipe No. IX gave the following data: Mean velocity = 0.111 ft. per sec.; loss of head = 0.12 ft. per hundred ft.; diameter = 0.03137 ft.; and temperature = 69.0° F. Find the coefficient of viscosity.

Substituting in (49.6) and taking g as 32, gives $\nu = \dfrac{h_f g\, D^2}{32\, L\, V}$ $= \dfrac{0.0012 \times 0.03137^2}{0.111} =$ 0.00001064 sq. ft. per sec. = 0.00988 sq. cm. per sec. Actually g is 0.5 per cent greater, making $\nu = 0.00993$ sq. cm. per sec. At 69° F. = 20.5° C., $\rho = 0.979$, and $\mu = 0.979 \times 0.00993 = 0.00972$ poises. For this temperature and distilled water, the best authorities give 0.995 centipoises. The discrepancy is only 2.3 per cent, which is within the experimental error for this experiment. In fact, the other part of the same pipe gave a loss of head of 0.13 ft. instead of 0.12 ft., which would increase the answer to $\dfrac{0.972 \times 0.13}{0.12} = 1.053$ centipoises, which is 5.8 per cent too large. These discrepancies illustrate the difficulty of working to high degrees of precision in many types of hydraulic experimentation. Had it been possible to take reliable measurements of the loss of head to thousandths of a foot per hundred feet, a better agreement of this result with such as are obtained in precise physical measurements might have been achieved. It is also to be noted as a factor in this example that the viscosity of the water that was used may have differed from that of distilled water.

PROBLEMS

49-1. Run No. 439 on Pipe No. XVI of the above series gave a velocity of 3.142 ft. per sec., a loss of head of 66.7 ft. per hundred ft., and a temperature of 38.5°. The diameter was 0.00893 ft. Find the kinematic viscosity in centistokes for this temperature.

Ans. 1.58. Distilled water gives 1.59 at this temperature.

49-2. A light oil flows through a standard 2-in. pipe with an average velocity of 6.3 ft. per sec. The loss of head is 11.43 ft. per hundred ft. What is the kinematic viscosity of the oil in the English and metric systems?

Ans. 0.000541 sq. ft./sec.; 0.503 stokes.

49-3. Oil with a kinematic viscosity of 0.005 sq. ft. per sec. flows in a 3-in. pipe at 10 ft. per sec. average velocity. Find the slope of the hydraulic gradient. *Ans.* 0.796

49-4. Oil with a kinematic viscosity of 0.0020 sq. ft. per sec. and weighing 58 lb. per cu. ft. is being pumped through a horizontal pipe with an inside diameter of 4 in. at an average velocity of 6 ft. per sec. What is the difference in pressure (in lb. per sq. in.) between two points 100 ft. apart?

Ans. 4.33 lb./sq. in.

49-5. What would have been the mean velocity in the example if the loss of head were still 0.12 ft. per hundred ft., but the temperature had been 32° F.? *Ans.* 0.061 ft./sec.

49-6. Show by calculus that the volume of a paraboloid of revolution is one-half the base times the altitude.

50. Reynolds Number.—Although the distinction between laminar and turbulent flow had been discovered by Hagen in 1839, a clear understanding of the conditions that determine which type of flow will exist awaited the work of Osborne Reynolds [3] (English, 1842–1912). From a consideration of the dimensions of the units involved, he came to the conclusion that the change from one type of flow to the other should come at some particular value of $\frac{V D \rho}{\mu}$. In 1880 he began experiments to check this supposition, using glass pipes ¼, ½, and 1 inch in diameter and about 4.5 feet long, fitted with trumpet-shaped mouthpieces. See Fig. 92. "The water was drawn through the tubes out of a large glass tank, in which the tubes were immersed, arrangements being made so that a streak or streaks of highly coloured water

[3] "An Experimental Investigation of the Circumstances Which Determine whether the Motion of Water Shall Be Direct or Sinuous, and of the Law of Resistance in Parallel Channels," *Phil. Trans. Royal Soc.*, Vol. 174, Part III (1883), p. 935; or *Scientific Papers*, Cambridge University Press (1900–1903), Vol. II, pp. 51–105.

entered the tubes with the clear water. The general results were as follows: (1) When the velocities were sufficiently low, the streak of colour extended in a beautiful straight line through the tube.

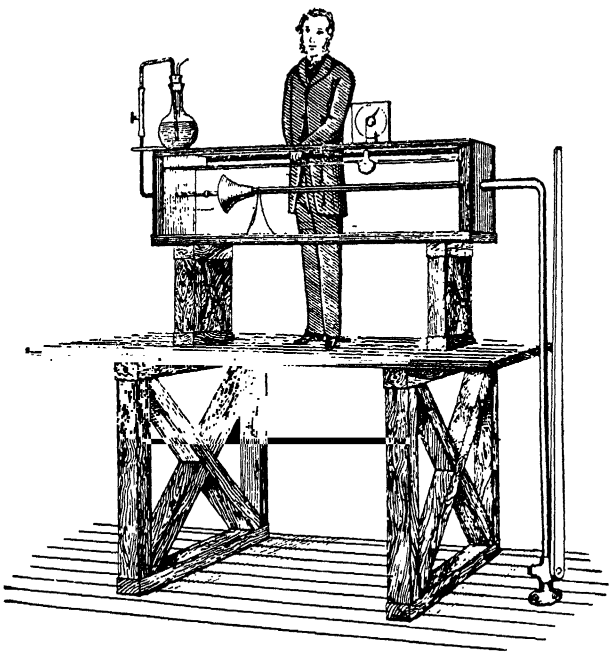


FIG. 92.—Osborne Reynolds' demonstration of critical velocity. (Taken from his original paper in *Phil. Trans. Royal Soc.*)

[Fig. 93 (a)]. (2) If the water in the tank had not quite settled to rest, at sufficiently low velocities, the streak would shift about the tube, but there was no appearance of sinuosity. (3) As the velocity was increased by small stages, at some point in the tube, always at a considerable distance from the trumpet or intake,[4] the colour band would all at once mix up with the surrounding water, and fill the rest of the tube with a mass of coloured water [as in Fig. 93 (b)]. On viewing the tube by the light of an electric spark, the mass of colour resolved itself into a mass of more or less

[4] The reason for this is that at the intake the velocity is equally distributed through the pipe, but as the drag of the walls slows up the outer layers, the center speeds up and reaches the critical velocity. In a sense, the true critical velocity is twice that given below, because $v_m = 2\ V$.

distinct curls, showing eddies [as in Fig. 93 (c)]." The velocity at which this occurred was found by measuring the rate of fall of the water surface in the tank. At different temperatures, and in the different pipes, this was found to be such as to give $\frac{V D \rho}{\mu}$ nearly the same value. The range was from 11,660 to 14,210, with an average of about 12,800. These figures include only experiments in which the water in the tank was initially at rest. If the water was in motion, the values were much smaller.

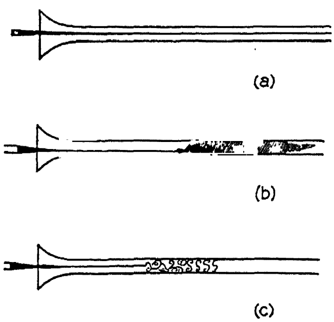


FIG. 93.—Laminar flow (a), and turbulent flow (b) and (c). (From Reynolds' original paper.)

Since Reynolds' time the expression $\frac{V D \rho}{\mu}$ has been found to enter into so many phases of fluid mechanics that it has been given a name and a symbol. It is universally called the *Reynolds number*. Different writers have used different symbols, but in this text it will be represented by **R**.

$$\mathbf{R} = V D \rho \quad \frac{V D}{\nu} \tag{50.1}$$

Since V is a length divided by time, D is a length, and ν is a length squared divided by time, **R** is dimensionless (an abstract number), and will be the same whatever the system of units used, provided they are consistent; that is, the same fundamental units are used for the three terms V, D, and ν.

In 1909, Ekman,[5] experimenting with the same apparatus as Reynolds used, showed that there is no definite upper limit for Reynolds' critical number, but that it depends on the lack of initial velocity in the entering water, and lack of vibration or disturbance in the pipe. By using special precautions, he got the Reynolds number up to 47,000 before the flow broke into turbulence. Using another pipe which was slightly larger at one end

[5] "On the Change from Steady to Turbulent Motion of Liquids" (in English), *Arkiv. Mat. Astron. Fysik*, Vol. 6 (1910), No. 12, pp. 1–16.

than at the other, he got up to 51,000, when the flow was from the large end to the small end, but only 44,000 when it was reversed. This is an illustration of the general fact that converging flow tends to be stable, and diverging flow, unstable.

Reynolds next experimented with two lead pipes, $\frac{1}{4}$ and $\frac{1}{2}$ inch in diameter, with a differential manometer attached, to show the pressure drop in 5 feet of the pipe. The water entered the pipe from the city water pipes through a fitting that would insure that the flow was turbulent. The water then flowed through 10.5 feet of the lead pipe before it entered the 5-foot test length. The flow was regulated by valves at both ends of the pipe. For the lower velocities, the discharge was caught in flasks of known capacity. For the higher velocities, the flow was measured through calibrated orifices. When the logarithms of the velocities were plotted against the logarithms of the pressure drops, it was found that the lower points lay on a 45° line (as the Hagen-Poiseuille law would require), and that the other points lay on another straight line, with a slope indicating that the exponent of the velocity was 1.723.

The highest point on the 45° lines corresponded to a Reynolds number of 2018 for the quarter-inch, and 2069 for the half-inch pipe. Similar experiments by later experimenters give results in about this same range, and the *lower critical* Reynolds number is generally taken as 2000. It is the value above which flow originally turbulent will remain turbulent; and below which flow originally turbulent will become laminar.

The results which Reynolds found for flow in lead pipes at velocities above the critical can be expressed in our notation by the formula

$$\frac{h_f}{L} = \frac{0.00524\, \nu^{2-n} V^n}{D^{3-n}} \tag{50.2}$$

where n in this case was 1.723. This can be rearranged into

$$\frac{h_f}{L} = 0.00524 \left(\frac{\nu}{D\,V}\right)^{2-n} \frac{V^2}{D} = 0.00524\, \mathbf{R}^{n-2} V^2 D^{-1}$$

This is the same as (29.1), if $f = \dfrac{0.00524 \times 2\,g}{\mathbf{R}^{2-n}}$. Taking

$g = 32.174$, this gives $f = \dfrac{0.337}{\mathbf{R}^{2-n}}$. This matter will be discussed further in the next article.

It may, however, be noted at this point, that the Hagen-Poiseuille formula can also be expressed in terms of the Reynolds number:

$$(50.3)\quad h_f = \frac{32\,\nu\,L\,V}{g\,D^2} = \frac{64\,\nu}{V\,D} \times \frac{L\,V^2}{2\,g\,D} = \frac{64}{\mathbf{R}} \times \frac{L\,V^2}{2\,g\,D}, \qquad \text{or}$$

$$(50.4)\qquad f = \frac{64}{\mathbf{R}}$$

It must be emphasized that this applies only to laminar flow. It may seem illogical to use a formula which assumes that the resistance varies as the square of the velocity for a case where we know that it varies as the first power. The answer is, that we are concealing a V in the denominator in our $\mathbf{R}$; that (50.4) is much easier to remember than (49.6); and that this makes (29.1) apply to all cases of pipe flow when the proper value of f is used.

EXAMPLE

Find the Reynolds number in the example of Art. 49.

The data gave $V = 0.111$ ft. per sec. and $D = 0.03137$ ft., and the solution showed $\nu = 0.00001064$ sq. ft. per sec. Then $\mathbf{R} = \dfrac{0.111 \times 0.03137}{0.00001064} = 327.$

PROBLEMS

50–1. Find the Reynolds number in Prob. 49–3. *Ans.* 500

50–2. Find the Reynolds number in Prob. 49–4. *Ans.* 1000

50–3. Find the Reynolds number in Prob. 49–2. Solve first in English units, and then check by converting the diameter and velocity to c.g.s. units.

50–4. Reynolds' smaller lead pipe had a diameter of 6.15 mm. The highest velocity for which the flow was steady (laminar) was 44.26 cm. per sec. The slope of the hydraulic gradient was 0.0516, and the temperature was 9° C., for which $\nu = 0.01349$ sq. cm. per sec. Check the value of $\mathbf{R}$ given in the text.

50–5. Reynolds' larger lead pipe had a diameter of 1.27 cm. The highest velocity for which the flow was steady was 22.60 cm. per sec. The temperature was 8° C., for which $\nu = 0.01387$ sq. cm. per sec. Check the value of $\mathbf{R}$ given in the text.

50–6. Compare the slope in Prob. 50–4 with that given by the Hagen-Poiseuille formula. Use $g = 980.7$ cm. per sec.

Ans. Formula gives 0.0515; therefore, error is less than 0.2%

50-7. What was the maximum velocity in Prob. 50-5? Where did it occur?

50-8. Check the derivation of equation (50.4).

50-9. Check the derivation of $f = 0.337\ \mathbf{R}^{-0.277}$.

50-10. Walker, Lewis, McAdams, and Gilliland, in their "Principles of Chemical Engineering," use a Reynolds number obtained by multiplying the diameter in in., by the velocity in ft. per sec., by the specific gravity, and dividing by the viscosity in centipoises. Find the number by which this must be multiplied to give the Reynolds number in consistent units.

Ans. 7742

51. Turbulent Flow in Smooth Pipes.—For many years the far-reaching implications of Reynolds' work were ignored by practical engineers, most of whom were entirely ignorant of it. Both mechanical and civil engineers continued to estimate the "friction loss" in pipes by methods that had been in use since Chezy. But the appearance of the new branches of Chemical Engineering, Petroleum Engineering, and Aeronautical Engineering, created a demand for methods which would apply to fluids other than water. The old style hydraulics men had apparently never considered whether or not (29.1) applied to fluids other than water. When experiments were tried, it was found that sometimes this formula applied and that sometimes it did not. It was only after some 30 or 40 years that Reynolds' work began to be studied by practical men.

But during these years, a few men had continued Reynolds' work and made some improvements. Ekman's extension of the higher critical number to 47,000 has already been mentioned. Blasius,[6] using the experiments of Saph and Schoder [7] on the flow of water in smooth brass pipes, having values of $\mathbf{R}$ as great as 104,500, found that for turbulent flow [8]

$$f = 0.316\ \mathbf{R}^{-0.25} \tag{51.1}$$

[6] "Das Ähnlichkeitsgesetz bei Reibungsvorgängen," *Physikalische Zeitschrift*, Vol. 12 (1911), p. 1175, or *Forschungsarbeiten* (*V.D.I.*), Heft 131 (1913).

[7] "An Experimental Study of the Resistance to the Flow of Water in Pipes," *Trans. A.S.C.E.*, Vol. 51 (1903), p. 253.

[8] The writer, not knowing of this work, recalculated the same data in 1914. The results are briefly summarized in *Trans. A.S.C.E.*, Vol. 101 (1936), pp. 709–712. In this connection, it may be noted that Reynolds' measurements on lead pipe have since been shown to be somewhat inaccurate, his values of f being several per cent too low. Actually, all smooth uniform pipe, whether of lead, brass, steel, glass, or wood, has, as nearly as can be measured, the same value of f for the range of Reynolds' numbers which he studied (up to 58,860).

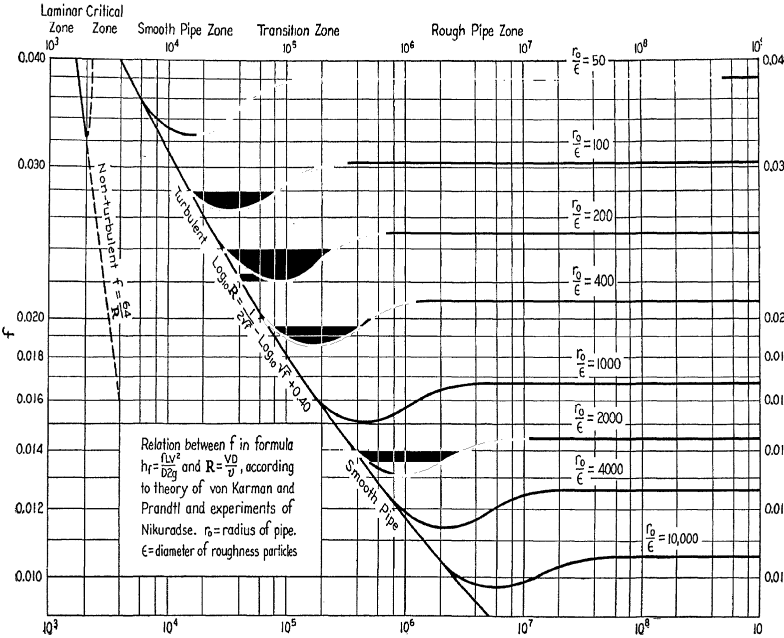


FIG. 94.—Relation between friction factor and Reynolds number.

Stanton and Pannell,[9] using air as the fluid, were surprised to find that roughening the walls of the pipe did not affect the resistance at the Reynolds numbers they were using. Later,[10] they made a more complete study, using both air and water, and using values of $\mathbf{R}$ up to 430,000, and showed that when $\log f$ is plotted against $\log \mathbf{R}$, the result is not really a straight line as assumed in (51.1), but a gradual curve, with values of $\log f$ decreasing as $\log \mathbf{R}$ increases, but at a decreasing rate (see Fig. 94). The first successful attempt to suggest a rational mathematical relationship between f and $\mathbf{R}$ for turbulent flow seems to have been made by von Karman [11] in 1930. This was simplified and improved by L. Prandtl [12] in 1933, and supplied with numerical constants by Nikuradse,[13] who carried his tests up to a Reynolds number of 3,230,000. The resulting formula, which it must be remembered applies only to very smooth pipes, is

$$\frac{1}{\sqrt{f}} = 2 \log_{10} (\mathbf{R} \sqrt{f}) - 0.80 \tag{51.2}$$

The theoretical basis on which this formula rests is discussed in Appendix C. It will be noted that if $\mathbf{R}$ is known, the equation cannot be solved directly for f. By assuming various values for f, however, a table or graph may be constructed, from which either f or $\mathbf{R}$ can be read if the other is known. If we wish, we can write the formula in the form

$$\log_{10} \mathbf{R} = \frac{1}{2\sqrt{f}} - \log_{10} \sqrt{f} + 0.40 \tag{51.3}$$

The range of values for which this formula applies will be discussed in the next article. It is plotted in Fig. 94.

Both Stanton and Pannell, and Nikuradse, made careful measurements of the velocity distribution and of the ratio of the mean

[9] "The Mechanical Viscosity of Fluids," by T. E. Stanton; *Proc. Royal Soc. (London)*, A, Vol. 85 (1911), pp. 366–376.

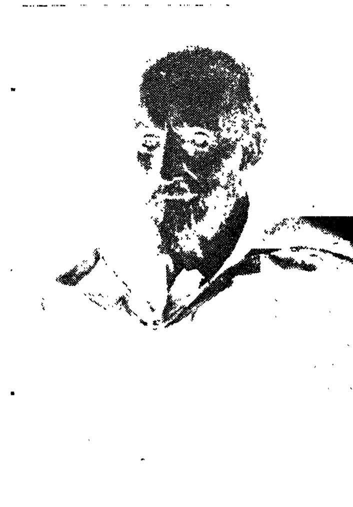
[10] "Similarity of Motion in Relation to Surface Friction of Fluids," by Stanton and Pannell; *Trans. Royal Soc. (London)*, A, Vol. 214 (1914), pp. 199–224.

[11] "Mechanische Ähnlichkeit und Turbulenz," translated into English as No. 611, *Tech. Mem., N.A.C.A.*, Washington, 1931.

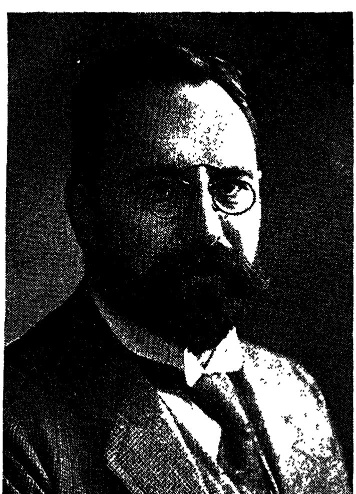
[12] "Neue Ergebnisse der Turbulenzforschung," *Z.d.V.D.I.*, No. 5 (1933).

[13] "Gesetzmässigkeiten der turbulenten Strömung in glatten Rohren," *V.D.I.*, Forsch. H. 356 (1932).

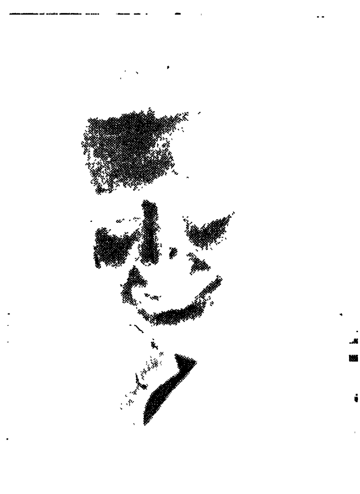
John R. Freeman (1855–1932)—American hydraulic engineer. Studied hydraulics of nozzles and jets. A patron of hydraulic research.



Osborne Reynolds (1842–1912)—English engineer and teacher. Developed theory of flow in pipes and hydraulic models.



Ludwig Prandtl (1875–)—German teacher of hydraulics. Developed theory of boundary layer.



Theodore von Kármán (1881–)—Born in Hungary, taught in Germany and the U. S. A leader in the study of turbulence.

to maximum velocity. They found that this depended on the Reynolds number, the average velocity approaching more nearly

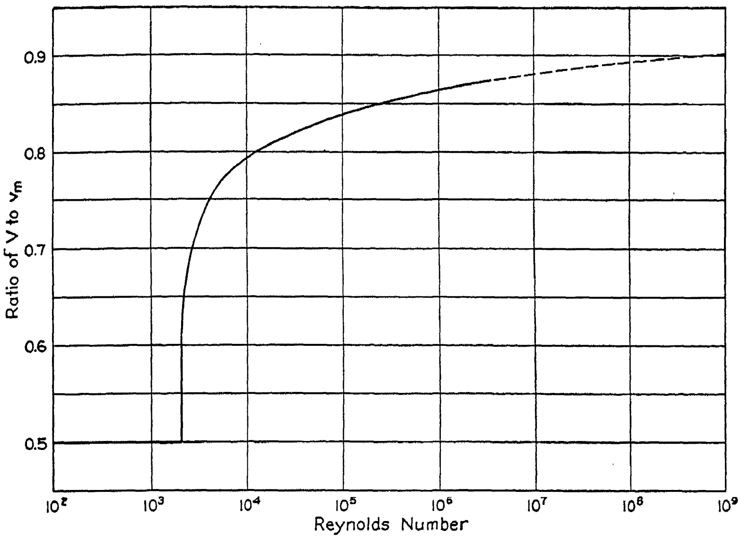


FIG. 95.—Ratio of average velocity to maximum velocity in smooth pipe.

to the maximum for high Reynolds numbers. Their results are shown in Fig. 95, which is based on one in Nikuradse's paper.[13a] (This is for smooth pipes.) Based on these values and Rehbock's approximate formula (39.3), the following values of the energy head correction factor, α, in smooth pipes, have been tabulated.

TABLE V.—VALUES OF α

R	$\frac{V}{v_m}$	δ	α
10,000	0.793	0.261	1.068
100,000	0.838	0.193	1.037
1,000,000	0.865	0.156	1.024
10,000,000	0.882	0.134	1.018
100,000,000	0.894	0.112	1.013

It should be understood that this velocity distribution is not attained at once when the fluid enters the pipe, but only after about 25 diameters of length have been traversed. In non-turbulent flow the required number of diameters is about one-tenth of the

[13a] See note 13 on page 175.

Reynolds number. The case of turbulent flow in rough pipes has already been discussed in Art. 39.

EXAMPLE I

For what values of the Reynolds number, if any, does equation (51.3) give the same f as does (51.1)?

The problem can be solved only by trial, or best by first plotting the two equations and seeing if they intersect. This was done, and there seem to be two intersections, at about $f = 0.038$, and at about $f = 0.020$.

If $f = 0.038$, $\frac{1}{2\sqrt{f}} = 2.565$, $\log \sqrt{f} = -0.710$, $\log \mathbf{R} = 3.675$ and $\mathbf{R} = 4732$. By (51.1), $f = 0.316 \times 4732^{-0.25} = 0.316 \div 8.298 = 0.03808$. If $f = 0.039$, $\frac{1}{2\sqrt{f}} = 2.532$, $\log \sqrt{f} = -0.704$, $\log \mathbf{R} = 3.636$, and $\mathbf{R} = 4325$. By (51.1), $f = 0.316 \times 4325^{-0.25} = 0.0390$. Therefore, one intersection is at $\mathbf{R}\quad 4325$. If $f = 0.020$, $\frac{1}{2\sqrt{f}} = 3.536$, $\log \sqrt{f} = -0.849$, $\log \mathbf{R} = 4.785$, and $\mathbf{R} = 60{,}950$. By (51.1), $\log \mathbf{R}^{0.2\cdot}\quad 1.196$, $10 + \log f = 9.500 - 1.196 = 8.304$, and $f = 0.02014$. If $f\quad 0.019$, $\frac{1}{2\sqrt{f}} = 3.627$, $\log \mathbf{R} = 4.888$, and $\mathbf{R} = 77{,}270$. By (51.1), $\log \mathbf{R}^{0.25} = 1.222$, $10 + \log f = 9.500 - 1.222 = 8.278$, and $f = 0.01897$. Therefore the other intersection is at about $\mathbf{R} = 74{,}000$.

EXAMPLE II

When $\log f$ is plotted against $\log \mathbf{R}$ as in Fig. 94, find the expression for the slope at any point and hence the value of n in an equation of the form of (51.1), which would fit for a limited range of values of $\mathbf{R}$.

Let $\log_{10} \mathbf{R} = x$, and $\log_{10} f = y$. Then $f = 10^y$, $\sqrt{f} = 10^{y/2}$, and equation (51.3) becomes $x = \frac{10^{-y/2}}{2} - \frac{y}{2} + 0.400$. Then differentiating with respect to x, $1\quad \frac{10^{-y/2}}{2}\left(-\frac{dy}{2\,dx}\right)\log_e 10 - \frac{dy}{2\,dx}$, $1 = -\left(\frac{2.3026}{4\sqrt{f}} + \frac{1}{2}\right)\frac{dy}{dx}$

or $\frac{dy}{dx}\quad \frac{-2}{1 + \frac{1.151}{\sqrt{f}}}$. In the equations derived from (50.2) f is a constant divided by $\mathbf{R}$ to the $2 - n$ power. That is, the slope of the line obtained by plotting $\log f$ against $\log \mathbf{R}$ will be $n - 2$. Substituting this for $\frac{dy}{dx}$ in the above, $2 - n\quad \frac{}{1 + \frac{1.151}{\sqrt{f}}}$, $2 + \frac{2.302}{\sqrt{f}}\quad \cdot n - \frac{1.151\,n}{\sqrt{f}} = 2$,

$n\sqrt{f} = 2.302 - 1.151\,n$, $0.869\,n\sqrt{f} = 2 - n$, and $n\quad \frac{}{1 + 0.869\sqrt{f}}$.

PROBLEMS

51–1. Find the value of **R**, when $f = 0.030$, by equations (51.1) and (51.3). *Ans.* 12,300; 11,200

51–2. Find the value of **R**, when $f = 0.010$, by equation (51.3). What value of f would equation (51.1) give for this **R**? What does this show regarding the use of the latter for large values of **R**? *Ans.* 2,510,000; 0.0079

51–3. Do the same as in the preceding problem for $f = 0.028$. *Ans.* 14,600; 0.0288

51–4. What value of n corresponds to equation (51.3) when $f = 0.010$? When $f = 0.040$? *Ans.* 1.840; 1.704

51–5. Drew, Koo, and McAdams in 1932 proposed the formula $f = 0.0056 + 0.500\ \mathbf{R}^{-0.32}$. Compare this with (51.3) at $\mathbf{R} = 4325$, at $\mathbf{R} = 77{,}270$, and at $\mathbf{R} = 3{,}000{,}000$.

Ans. (51.3) gives $f = 0.0390$, 0.0190 and 0.0097; this formula gives 0.0399, 0.0192, and 0.0098

51–6. Nikuradse proposed the empirical formula $f = 0.0032 + 0.221\ \mathbf{R}^{-0.237}$ as a substitute for (51.3). Compare them for the three values of **R** used in Prob. 51–5. Which is better, this formula or the one given in the preceding problem?

Ans. This formula gives 0.0336, 0.0185, and 0.0096, as compared with 0.0390, 0.0190, and 0.0097 for (51.3)

51–7. A certain fuel oil weighs 58 lb. per cu. ft. and at a temperature of 60° F. has a kinematic viscosity of 0.020 sq. ft. per sec. Find the power required to pump it through 1000 ft. of very smooth 6-in. pipe at a mean velocity of 5 ft. per sec., if the efficiency of the pump is 80 per cent. *Ans.* 51.5 hp.

51–8. What power would be required in the preceding problem if the temperature is raised to 150° F. at which the kinematic viscosity is 0.00052 sq. ft. per sec.? *Ans.* 3.82 hp.

51–9. If at a temperature of 120° F. the oil in the two preceding problems had a kinematic viscosity of 0.00125 sq. ft. per sec., what power would be required at that temperature? *Ans.* 3.22 hp.

51–10. For what Reynolds number is the slope of (51.3) plotted logarithmically, a minus one-quarter, as in (51.1)? *Ans.* 16,700

52. Boundary Layer.—The whole subject of turbulent flow has been greatly clarified by an idea introduced by L. Prandtl in 1904. Suppose that a flat plate, as in Fig. 96, moves edgewise through a liquid at rest. The liquid outside the surface represented by the line ABC (and a similar line on the other side of the plate, not drawn in the figure) will be practically undisturbed. If there is any disturbance, it will be almost independent of the viscosity of the liquid; that is, it will be due to inertia forces only. Near the plate, however, there will be a narrow region inside the line ABC,

called the boundary layer, within which the velocity varies rapidly as we approach the plate, so that the shearing forces are large, and the viscosity of the liquid has a great deal to do with what hap-

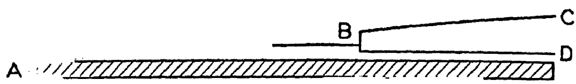


Fig. 96.—Diagram of boundary layer.

pens. Of course, there is no sharp line of division between the two regions, but it is necessary to assume one to simplify our problems.

To take a numerical case, suppose that the plate in Fig. 96 is moving toward the left at a velocity of 1 meter per second, and that the liquid is water at room temperature ($\nu = 1$ centistoke). Prandtl [14] showed from hydrodynamical theory that the thickness of the boundary layer at a distance x from A would be approximately

$$\delta = C\sqrt{\frac{\nu\, x}{V}} \tag{52.1}$$

where V is the velocity of the plate relative to the undisturbed liquid. Blasius showed that for a flat plate $C = 3.4$. For the case assumed above, $\nu = 0.01$, $V = 100$, and $\delta = 0.034\sqrt{x}$. That is, at 1 centimeter from A, the boundary layer would have a thickness of 0.034 centimeter = 0.34 millimeter, and at a meter from A it would be ten times as thick, or 3.4 millimeters. Actually, when we get far enough from A so that $\dfrac{V\,\delta}{\nu}$ is about 3000 (in this case $\delta = 0.3$ centimeter or $x = 78$ centimeters), the law changes. This is point B in the figure. The flow in the boundary layer becomes unstable and breaks into turbulent flow (excepting for a very thin laminar sub-layer, which remains inside the turbulent boundary layer). The thickness of the turbulent boundary layer on a flat plate (including the laminar sub-layer) has been found to be

$$\delta = 0.37\left(\frac{\nu}{V}\right)^{0.2} x^{0.8} \tag{52.2}$$

[14] See Chapter IV, "Boundary Layers," in *Applied Hydro- and Aeromechanics*, by L. Prandtl and O. G. Tietjens, McGraw-Hill Book Co. (1934).

These facts help us to understand more clearly what takes place in pipes. When fluid enters a pipe it at first has practically uniform velocity (over the whole section if the entrance is rounded, or over the contracted vein for an abrupt entrance). But a boundary layer at once begins to form along the wall of the pipe in which the velocity varies approximately as the dis-

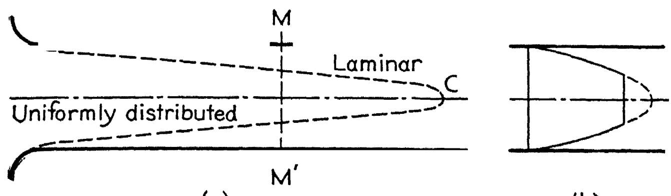


FIG. 97.—Velocity distribution near entrance, low **R**'s.

tance from the wall (actually the variation is parabolic as we found in Art. 49). If the pipe is small enough, or the flow small enough, or the kinematic viscosity great enough, so that **R** for the pipe does not exceed 2000 or a little more, the thickness of the boundary layer will steadily continue to increase as we move downstream until it reaches the center of the pipe, and the whole flow is of the sort described in Art. 49. This will occur at x = about 0.1 **R** D. The approximate situation when **R** = 40 is shown in Fig. 97. Fluid inside and to the left of the dashed line is moving with a velocity which is uniform across the section

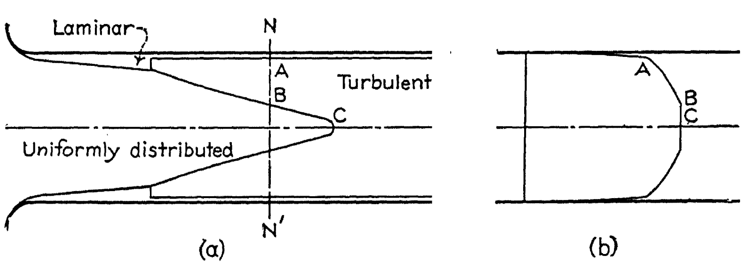


FIG. 98.—Velocity distribution near entrance, high **R**'s.

(but constantly increasing downstream); that outside and to the right of the dashed line is flowing in the manner described in Art. 49. A velocity distribution diagram for section MM' is shown in Fig. 97 (b). It must be realized that these figures are diagrammatic only.

When the Reynolds number of the pipe flow is above critical, the situation is as shown by Fig. 98 (here the scale is very much distorted, so as to make the various parts visible). Outside of A is the laminar sub-layer in which the velocity varies nearly in proportion to the distance from the wall (since the parabola is nearly straight here); between A and B is the turbulent boundary layer; and between B and C the velocity is uniform. Fig. 98 (b) is a velocity distribution diagram for section NN'. As stated before, it seems to require about 25 diameters of length for the turbulent boundary layer to reach the center of the pipe; but the experimental data are not conclusive, and the true value quite likely depends on the Reynolds number. Figure 99 shows three velocity distributions for a pipe of the same size, and with the same v_m, at a point far enough from the entrance so that there is no further change; that is, downstream from point C in Figs. 97 and 98. The inner curve is the parabola showing the distribution when the flow is laminar (due to high viscosity); the outer curve is for a high Reynolds number in a smooth pipe; and the other is for a very rough pipe and a high Reynolds number. The data are from Nikuradse.

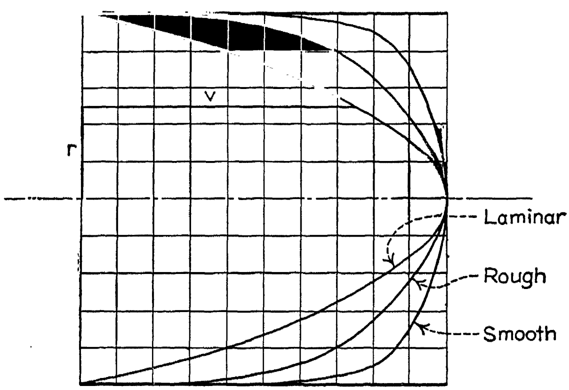


Fig. 99.—Velocity distribution far from entrance.

EXAMPLE

Rouse gives the following formula for the thickness of the laminar sub-layer: $\delta = \dfrac{8\, v_w D}{V \sqrt{f}\, \mathbf{R}}$. v_w is the velocity at the boundary between the laminar and turbulent portions. Assuming that $v_w = 0.5\, V$, find the thickness of the

boundary layer for a smooth pipe 1 ft. in diameter carrying a fluid whose kinematic viscosity is 1 centistoke at a mean velocity of 10 ft. per sec.

$\mathbf{R} = \dfrac{10 \times 1 \times 929}{0.01} = 929{,}000$. For this, (51.3) or the curve of Fig. 94, gives $f = 0.0118$. Then $\delta = \dfrac{8 \times 0.5 \times 1}{0.0118 \times 929{,}000} = 0.000365$ ft. or, say, 0.0044 in.

PROBLEMS

52–1. For the flat plate mentioned above (V = 100 cm. per sec., ν = 0.01 sq. cm. per sec.) find the thickness of the boundary layer at 78 cm. from the cutting edge by (52.1). *Ans.* 3.0 mm.

52–2. Solve the preceding problem by (52.2). If these formulas are correct, what does this show occurs at point B in Fig. 96? *Ans.* 19.1 mm.

52–3. Solve the preceding two problems for x = 100 cm. *Ans.* 0.34 cm.; 2.34 cm.

52–4. Estimate the thickness of the boundary layer at the downstream end of a bridge pier which is 40 ft. long when the water is flowing at 10 ft. per sec. Take the temperature as 68° F. *Ans.* 0.45 ft.

52–5. By solving (52.1) and (52.2) simultaneously, find for what value of $\dfrac{V\delta}{\nu}$ turbulent flow might be expected to begin. *Ans.* 137

52–6. Using the formula given in the example, the same 1 ft. pipe, and the same assumption that v_w is 0.5 V, find the thickness of the laminar sublayer when $\mathbf{R}$ = 4325. What mean velocity will give this Reynolds number if ν is 1 centistoke? *Ans.* 0.28 in.; 0.0466 ft./sec.

53. Flow in Rough Pipes.—The work of Nikuradse[15] on flow in rough pipes has already been referred to in Chapter IV. The values of f which he found are shown in Fig. 100. The first point to notice is that the four pipes with the least roughness follow the line representing smooth pipes very closely for some distance. Another point to notice is that at high enough Reynolds numbers all the curves become horizontal straight lines. By using the dimensionless numbers $\dfrac{\epsilon}{\nu}\sqrt{\dfrac{\tau_0}{\rho}}$ and $\dfrac{1}{\sqrt{f}} - 2\log_{10}\dfrac{r_0}{\epsilon}$, Nikuradse brought his results into the comparatively simple relationship shown in Fig. 101. τ_0 is the unit shearing stress on the walls of the pipe, which he had taken sufficient measurements to determine. The form of the dimensionless numbers was not based on surmise, but was suggested by the theoretical work of Prandtl and von Karman, as is explained further in Appendix C.

[15] "Strömungsgesetze in rauhen Rohren," *Verein Deutscher Ingenieure*, Forschungsheft 361 (1933).

Figure 101 is based on relative roughnesses (or rather relative smoothnesses, $\frac{r_0}{\epsilon}$) between 15 and 507, and there is, of course, some question whether it will apply at higher values of $\frac{r_0}{\epsilon}$. But as there seems to be no apparent reason why it should not, the

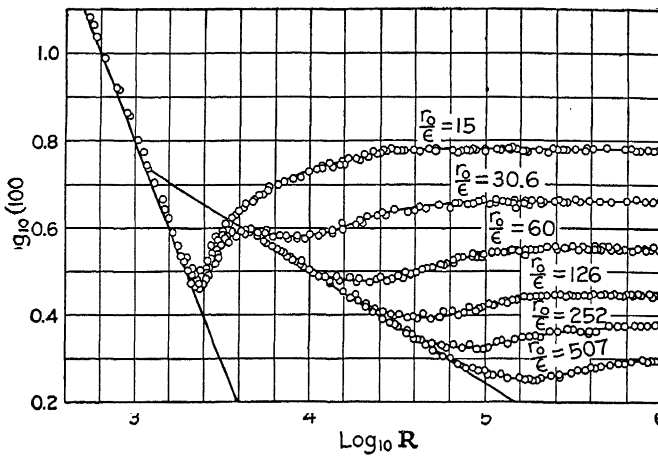


FIG. 100.—Nikuradse's experiments on rough pipe.

writer has assumed that it holds for all values of $\frac{r_0}{\epsilon}$, and has constructed Fig. 94 to forecast the value of f for all values of $\mathbf{R}$ and $\frac{r_0}{\epsilon}$. It will be seen that there are five distinct zones, depending on the value of Reynolds number. (1) The laminar zone, where $\mathbf{R}$ is less than 2000, $f = \frac{64}{\mathbf{R}}$, and the head lost varies as the first power of the velocity. (2) The critical zone, where $\mathbf{R}$ is between 2000 and, say, 3000. Here the flow may be laminar, but is more likely to be turbulent. There is some evidence that the head here varies as the cube of the velocity as shown by the dashed ascending line in Fig. 94, but it is safest to regard it as an unpredictable zone. (3) The smooth pipe zone where the flow is turbulent but f depends only on the Reynolds number and not at all on the roughness of the pipe wall. (4) A transition zone where f depends mostly upon the relative roughness, but still somewhat upon the Reyn-

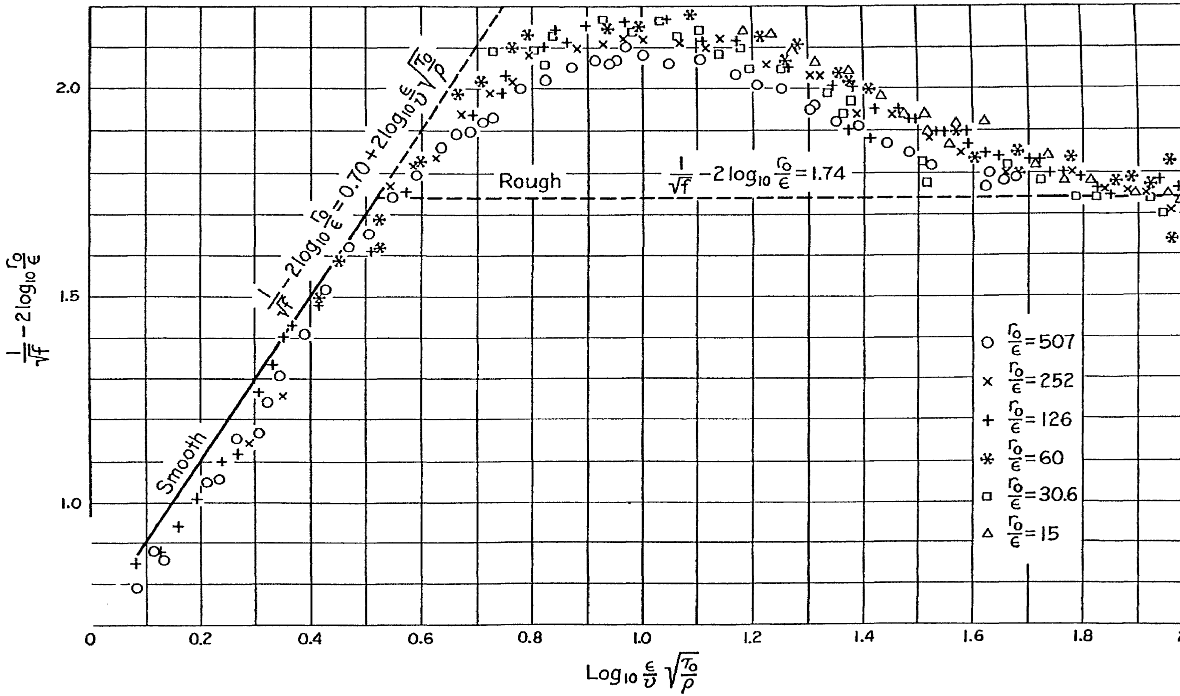


Fig. ing $\frac{1}{\sqrt{f}} - 2$;ainst $\log \frac{\epsilon}{v}\sqrt{\frac{\tau_0}{\rho}}$

olds number. (5) The rough pipe zone, where f depends entirely on the relative roughness and not at all on the Reynolds number.[16]

The dividing lines between zones (3), (4), and (5) are not at fixed values of **R**, but depend on the relative roughness. As will be seen from Fig. 94, the value of f at which the roughness begins to affect the result, is practically the same as the value of f for very high Reynolds numbers. This is also shown in Fig. 101 by the intersection of the two straight lines. Therefore, the **R** which divides zone (3) from zone (4) can be found by solving equations (30.1) and (51.3) simultaneously. It will be found that for values of $\frac{r_0}{\epsilon}$ less than 24 there will be no zone (3), and that zones (2) and (4) merge. This is illustrated by the two upper curves in Fig. 100. By inspection of Figs. 94, 100 and 101 it will be seen that the dividing line between zones (4) and (5) comes at about 18 times as large a Reynolds number as that between (3) and (4).

There is a physical explanation for the phenomenon of a rough pipe acting as a smooth one in zone (3). Nikuradse showed that the thickness of the laminar film in pipes is

$$\delta = \frac{11.6\,D}{\mathbf{R}\sqrt{f/8}} \tag{53.1}$$

Solving this simultaneously with (51.3) gives

$$\log\frac{r_0}{\delta} = \frac{}{2\sqrt{f}} - 1.417 \tag{53.2}$$

Solving this simultaneously with equation (30.1), gives

$$\log\delta = 0.547 + \log\epsilon, \qquad \text{or} \qquad \epsilon = 0.284\,\delta$$

That is, if the average height of the roughness particles is not over 28 per cent of the thickness of the laminar sub-layer, they are entirely imbedded in it, and have no effect on the main flow in the pipe. (The flow in the laminar sub-layer itself is only a

[16] Colonel Du Buat (French, 1734–1809), the "father of the hydraulic laboratory," in his *Principes d'Hydraulique Vérifiés par un Grand Nombre d'Expériences* (1779) argued that the resistance to the flow in pipes was independent of the roughness of the walls because the water filled the voids and the only effective friction was that of water upon water. For this "fallacy" he was criticized for 150 years, but now we realize that he had seen a facet of truth that had escaped others. In zones (1), (2), and (3) he was right; in zones (4) and (5) he was not.

negligible per cent of the whole flow.) If the average height of the roughness particles becomes more than 28 per cent of the thickness of the sub-layer, some of the roughness particles extend far enough so that the stream lines in going around them interfere with the turbulent flow, extra eddies are set up, and the resistance to flow increases. The situation is shown diagrammatically in Fig. 102.

The fact that at sufficiently high Reynolds numbers (in zone 5) the resistance is independent of the Reynolds number, and hence of the viscosity, is the reason we could work many problems in pipe flow (as in Chapter IV) without knowing anything about viscosity. As will be seen in the next chapter, this is but one example of many in which at large Reynolds numbers the phenomena become independent of the Reynolds number.

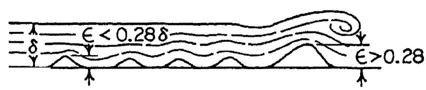


FIG. 102.—Diagram of roughness.

There is another sort of roughness shown diagrammatically in Fig. 103. This is called *waviness*. It does not produce a resistance such as Nikuradse found for his roughened pipes. Instead, f continues to decrease for increasing $\mathbf{R}$ along a line about parallel to the smooth pipe line, but at values of f from as much as 20 to 30 per cent higher. Also, tests on fairly smooth pipe lines, such as are used in transporting crude oil, natural gas, etc., give values of f from 5 per cent to 10 per cent above those given by (51.3). This may be partly due to the presence of minor losses caused by slight changes of section, bends, etc., and partly to slight waviness in the pipe walls. As yet the rough pipe curves of Fig. 94 can hardly be used in engineering work, because the proper values of ϵ for various kinds of pipes are not accurately known. Furthermore, whether this is the proper kind of curve to use is still doubted by some whose opinion must be respected.

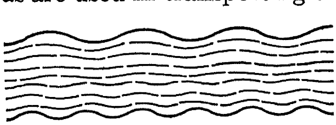


FIG. 103.—Diagram of waviness.

PROBLEMS

53–1. If $\frac{r_0}{\epsilon} = 44$ and Reynolds number is high, find the value of f.

Ans. 0.0396

53–2. Find the value of $\mathbf{R}$ for which a smooth pipe has this same value of f.

Ans. 4110

53-3. By solving (30.1) and (51.3) simultaneously, show that the Reynolds number which divides zone (3) from zone (4) is given by the equation $\log \mathbf{R} = 1.57 + \log \frac{r_0}{\epsilon} + \log \left(0.87 + \log \frac{r_0}{\epsilon}\right)$.

53-4. Check the derivation of (53.2).

53-5. Using (53.2) and (30.1), show that $\epsilon = 0.284\,\delta$.

53-6. The largest pipe known to have been tested was the Ontario Power Tunnel, reported by F. C. Scobey. It was 18 ft. in diameter and when $V = 20$ ft. per sec. gave a slope of 2.397 ft. per 1000 ft. Assuming the temperature as 55° F., what was the Reynolds number? By what per cent does the observed f differ from that given by equation (51.3)?

Ans. 27,500,000; − 1.3%

53-7. Solve the preceding problem, assuming the temperature to be 61.7° F. *Ans.* 30,300,000; 0.0%

53-8. A test on woodstave pipe 13.5 ft. in diameter, made by F. C. Scobey, gave $V = 6.09$ ft. per sec. and $S = 0.000452$. Assuming the temperature as 55° F., find **R**. By what per cent does the observed f exceed that given by (51.3)? *Ans.* 6,290,000; 22.0%

53-9. In 36-in. pipe for which $\epsilon = 0.01$ ft., at what velocity would water at 68° F. cease to act as if the pipe were smooth? *Ans.* 0.0617 ft./sec.

CHAPTER VII

EFFECT OF VISCOSITY

54. General Considerations.—Since shear in a liquid is proportional to the rate of change in velocity at right angles to the flow, when the velocity is everywhere zero, the shear is zero, whatever the viscosity. Therefore, the statements of Chapter I are true of all liquids whatever their viscosity.

The same applies in general to Chapter II. However, although viscosity has no direct effect, it has some indirect effects. By reason of viscosity the outer layers of liquids flowing in contact with fixed walls are retarded, so that the velocity distribution becomes non-uniform. How this effect is corrected for by introducing α into energy equations and β into momentum equations has already been explained in Art. 39.

When we come to Chapter III (Orifices, Tubes, Nozzles, and Weirs), we find that the omission of viscosity considerations makes our former treatment incomplete. If the kinematic viscosity is very small, Reynolds number is very large. We have found that in pipes, if Reynolds number is large enough, it no longer has any effect.[1] The same is found to be true for orifices, weirs, etc. Therefore, the values of constants given in Chapters III and IV are in general true for water, but would not be true for a viscous liquid like lubricating oil. This, however, is not the whole story. Reynolds number depends on velocity and size as well as on viscosity. Therefore, if the product of velocity times size is large enough, the values of coefficients previously given may be correct even for viscous liquids. On the other hand, if the product of velocity times size is small enough, the coefficients may have to be changed, even for water. This is the reason of the limitations noted in Arts. 20, 22, 25, etc.

55. Effect of Viscosity on Orifice Flow.—The value of 0.60

[1] This is not the same as saying that the liquid behaves as a perfect fluid of zero viscosity. Viscosity has an important effect, but its *magnitude* is not significant.

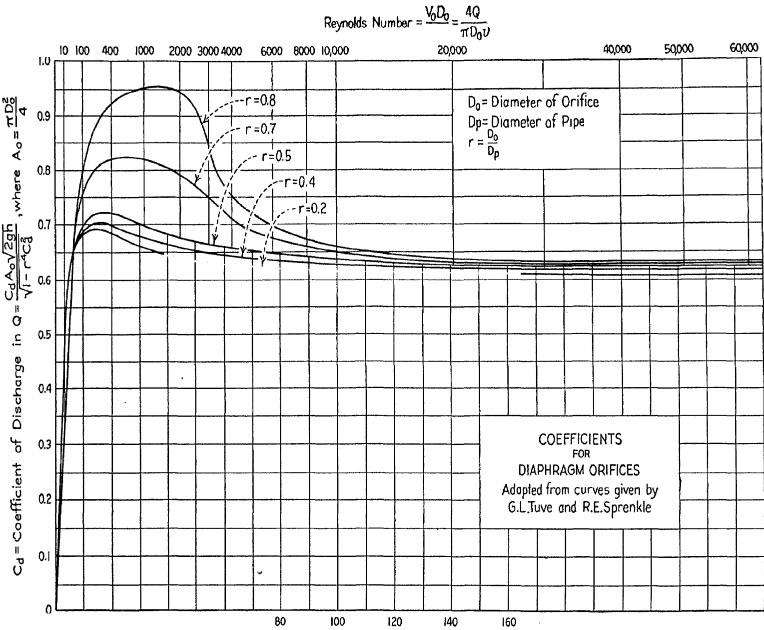


Fig. 104.—Coefficients for diaphragm orifices.

given in Art. 20 as the coefficient of contraction (and of discharge) for sharp-edged orifices is fairly accurate as long as Reynolds number is, say, 30,000 or more. With an increase of viscosity, or, what amounts to the same thing, a reduction of size or velocity so that the existing viscosity has more effect, two things happen which have opposite tendencies. First, the flow along the orifice plate toward the orifice is slowed up. This tends to remove the cause for contraction, and the jet becomes larger. At first this is the more important effect, and the coefficient of discharge increases with a decrease in the Reynolds number. See Fig. 104, which is based on measurements by Tuve and Sprenkle.[2] Generally at a Reynolds number somewhere between 180 and 1400, depending on the ratio of orifice diameter to pipe diameter, the discharge coefficient reaches a maximum. (The figures just given refer to orifice diameters of 0.2 and 0.8 the diameter of the pipe, respectively. Any reduction below 0.2 changes the curve very slightly, but any increase above 0.8 would make a great difference, and the maximum would come at a value of **R** above 1400. The maximum value of C_d would not be over 1.00, however.)

As **R** decreases still further, viscosity reduces the velocity of the jet more than enough to make up for any further increase in

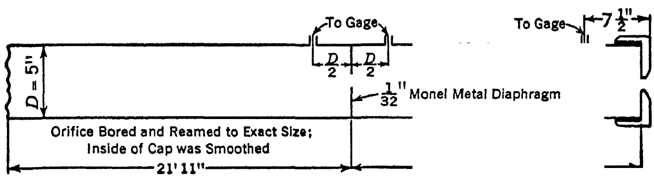


FIG. 105.—Diaphragm orifice at center, cap orifice at right. Used by Professor Horace Judd in his 1916 experiments. (This and Figs. 54 and 106, are from a paper "Discharge Coefficients for Pipe Orifices," by Professor Wallace M. Lansford in *Civil Engineering,* May, 1934, pp. 245–247.)

size, so that C_d is reduced. In fact, the jet has probably already become practically as large as the orifice and can increase no more. Most of the experiments have been made on diaphragm orifices such as that in Fig. 106 and at the center of Fig. 105, so that the

[2] "Orifice Coefficients for Viscous Liquids," *Instruments;* Vol. 8, Nos. 8 and 9 (Aug. and Sept., 1935), pp. 202–205, 225, 232–234. See also *The Flow of Water through Orifices,* by S. R. Beitler, Bull. No. 89 of Engineering Experiment Station, Ohio State University (1935); and *Fluid Mechanics for Hydraulic Engineers,* by Hunter Rouse; McGraw-Hill Book Co. (1938), pp. 256–262.

jet has not been actually measured. But since the coefficients of discharge obtained are practically the same as those obtained with a free jet, it is assumed that the submergence of the jet has very little effect on its size.

For Reynolds numbers less than 10, the jet expands quickly and smoothly to fit the downstream pipe, the outer stream lines closely following the downstream side of the orifice and the pipe wall. For this case we find that C_d varies directly as the square root of the Reynolds number. In Fig. 104, the scale of **R** has been taken so that $\sqrt{\mathbf{R}}$ is an even scale; this makes the lines straight up to $\mathbf{R} = 10$. Then

$$C_d = \sqrt{k\,\mathbf{R}} \tag{55.1}$$

When the velocity of approach can be neglected,

$$Q = C_d \frac{\pi D_o^2}{4} \sqrt{2\,g\,h}, \tag{55.2}$$

where h is the differential head. Dividing through by the area of the orifice, $V_o = \dfrac{Q}{A_o} = C_d \sqrt{2\,g\,h}$. Eliminating V_o between this and (55.1) gives $C_d = k\,D_o\,\sqrt{2\,g\,h}$, and substituting this in (55.2) gives

$$Q = \frac{\pi\, g\, k\, D_o^3 h}{2\,\nu} \tag{55.3}$$

That is, in this range the discharge is proportional to the first power of the head and to the cube of the diameter, and inversely proportional to the kinematic viscosity. This is somewhat analogous to the case of laminar flow in pipes. It is doubtful, however, whether it has any practical application.

In numerical work with curves like Fig. 104, the student must be on his guard. Reynolds number for a diaphragm orifice might be, and sometimes is, based on the diameter of the pipe and the velocity in the pipe. Or it might be based on the diameter and velocity of the jet at the *vena contracta*. In the curves presented here, $\mathbf{R} = D_oV_o$ where D_o is the diameter of the orifice, and V_o

is the discharge divided by the area of the orifice; that is, it is the average of the forward components of the velocities in the plane of the orifice. The C_d here plotted is the product of C_v and C_c for the contracted vein. As shown in Prob. 55–2 below, this

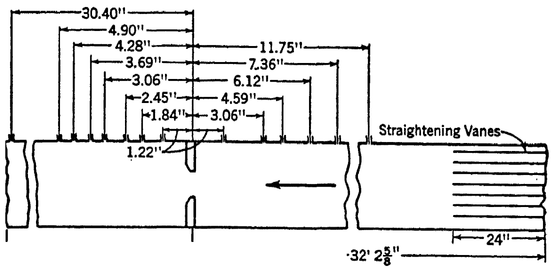


FIG. 106.—Diaphragm orifice used by Beitler and Bucher in their study of the best location for pressure taps.

is different from the C used by the A.S.M.E. Special Research Committee on Fluid Meters. (Professor Daugherty, in his *Hydraulics*, uses the symbols C and C_d, but with the meanings interchanged.)

There is much to be said for a constant which includes the velocity of approach correction, such as

$$Q = K A_o \tag{55.4}$$

where A_o is the area of the orifice. Or, for any one orifice, this can be still further simplified to

$$Q = K' \sqrt{h} \tag{55.5}$$

K' varies somewhat with the head; but so do K, C, and C_d.

There is one phenomenon that is present in the free orifice but not in the submerged; namely, surface tension. According to H. W. Swift,[3] the effect of surface tension can be corrected for approximately by subtracting $\dfrac{0.008}{D}$ from the observed head. If this is true, for values of D of one inch or more the error due to neglecting surface tension will not be more than that due to an

[3] "Orifice Flow as Affected by Viscosity and Capillarity," *Phil. Mag.*, Vol. 2 (1926), pp. 852–875.

error of 0.008 inch or 0.0007 foot in the head. Such errors could be neglected in most cases. For small orifices, particularly at low heads, this effect must be considered.

EXAMPLE

In Art. 20, the limiting values were set at $D = 1$ in., $H = 2$ ft., and viscosity equal to that of water. To what Reynolds number is this equivalent?

Since we are interested in the minimum **R**, use the maximum ν of water, which is 0.00001931 sq. ft. per sec. The ideal velocity corresponding to a 2-ft. head is $8.02\sqrt{2} = 11.34$ ft. per sec. This is the velocity in the jet. The average forward component of the velocities in the plane of the orifice, $\frac{Q}{A_o}$, will be 0.60 of this, or 6.80 ft. per sec. Then

$$\frac{6.80 \times 1}{12 \times 0.00001931} = \frac{6.80}{0.0002317} = 29{,}400.$$

PROBLEMS

55–1. If the diameter of the pipe $= D_p$, the diameter of the diaphram orifice $= D_o = r\,D_p$, the area of the orifice $= A_o$, the area of the jet $= A_j$ $= C_c A_o$, and the differential head $= h$, show that $Q = \frac{C_d A_o \sqrt{2\,g\,h}}{\sqrt{1 - }}$.

55–2. If $Q = \frac{}{\sqrt{1 - r^4}}$ as in (12.4), derive an equation for C_d in terms of C and r.

55–3. Using the curves of Fig. 104, estimate the flow of water at 68° F. through a 1-in. diaphragm orifice in a 2-in. pipe, if the pressure drop is 0.09 ft. of water. (It will be necessary to solve by successive approximations, assuming a value of C_d as a first approximation.) *Ans.* 0.00837 c.f.s.

55–4. Solve the preceding problem if the temperature is changed to 32° F. Then if it is changed to 104° F. *Ans.* 0.00860 c.f.s.; 0.00830 c.f.s.

55–5. What would be the kinematic viscosity of another more viscous fluid which would give the same discharge as in Prob. 55–3, except that the 0.09 ft. head would be measured by a column of that fluid?

Ans. 0.0034 sq. ft./sec.

55–6. Estimate the flow of air at 68° F., through a 2-in. orifice in a 4-in. pipe, if the absolute pressure just upstream from the orifice is 35.0 ft. of water, and the absolute pressure just downstream from the orifice is 34.0 ft. of water. (Air at atmospheric pressure and this temperature is 1/830 as heavy as water.) *Ans.* 3.19 c.f.s.

55–7. Heavy California crude oil flows through a 1-in. orifice in a 12-in. pipe. If the pressure drop is 0.506 ft., what is the discharge?

Ans. 0.0156 c.f.s.

56. Effect of Viscosity on Nozzles and Venturis.—For Reynolds numbers greater than about 150,000 (based on the velocity and diameter at the throat), viscosity has little effect on the coefficient of discharge of nozzles. As stated in Art. 23, C_d will vary from about 0.95 to 0.98 or more, depending on the proportions of the nozzle, its inside finish, and its size. Of two nozzles of the same proportions and inside finish, the larger will have the greater C_d because of the reduced relative roughness. Experiments on the flow of viscous fluids through nozzles do not seem to have been published, but there is every reason to suppose that coefficients would vary in much the same way as in the Venturi meter discussed below.

A form of submerged nozzle called the *flow nozzle* is coming more and more into use as a means of measuring the flow in pipes. The arrangement is shown diagrammatically in Fig. 107. It has much the same properties as a Venturi meter, and is cheaper and simpler to install, being inserted in a flange coupling the same as a diaphragm meter. The head lost after passing the throat is greater than in the Venturi meter. If h is the differential pressure between the two taps shown in Fig. 107, h_f is the total or over-all pressure loss from the upstream tap to a point about four diameters downstream, and r is the ratio of throat diameter to pipe diameter, it has been found [4] that

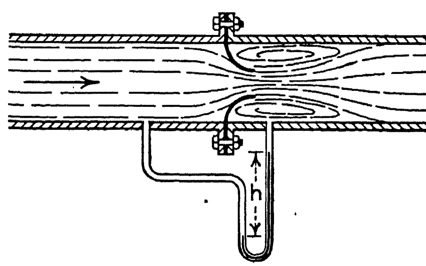


FIG. 107.—Diagram of flow nozzle.

$$h_f = (1 - r^2)\,h \tag{56.1}$$

This means that if the throat diameter is half the pipe diameter, one-quarter of the differential head is recovered as the jet expands to full pipe size downstream from the nozzle, and three-quarters is lost. If the throat diameter is 0.80 of the pipe diameter, 0.64 h will be recovered and h_f = 0.36 h. For detailed information

[4] For the latest information see "Some Results from Research on Flow Nozzles," by H. S. Bean and S. R. Beitler, *Trans. A.S.M.E.*, RP-60-3 (April, 1938), pp. 235–244.

as to the value of the coefficient of discharge in these nozzles, the student is referred to the paper just mentioned.[4a] In general the coefficient varies in much the same way as that of the Venturi meter shown in Fig. 108.

The formula for the flow through a Venturi meter (12.4) was derived in Art. 12. It applies also to the flow in a nozzle or flow nozzle. Tests on Venturi meters have been extended to oils as

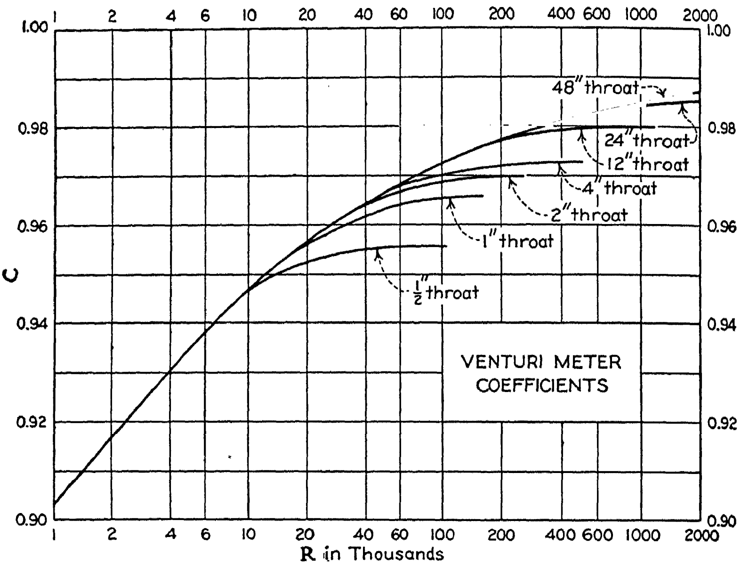


FIG. 108.—Coefficients for Venturi meters.

well as water, air, and various other fluids, so that a very wide range of Reynolds numbers has been covered. The results of these experiments are not entirely consistent but in general they follow the pattern of Fig. 108, which is based on tests which were made by Ed S. Smith[5] in 1923 on Simplex Standard Venturis of cast iron, with $r = 0.5$. He also made tests on a glass Venturi of about the same proportions down to Reynolds numbers of about 0.01, when C was only 0.011. From here up to about $\mathbf{R} = 6$, $\log C$ plotted against $\log \mathbf{R}$ gave a straight line with a slope of

[4a] See note 4 on page 195.

[5] "The Oil Venturi Meter," *Trans. A.S.M.E.*, Vol. 45 (1923), pp. 67–75. For more recent work see "Fluid Meters, Their Theory and Applications," *A.S.M.E.* research publication, 4th Ed. (1937), Part 1.

one-half, showing C to be proportional to the square root of the Reynolds number, as we found in the previous article that it was for orifices at correspondingly low Reynolds numbers.

For Reynolds numbers above this range, the coefficient increases less rapidly up to an **R** of between 10,000 and 1,000,000 (the particular value depending on the size of the throat), where C begins to be different for different sized throats. At a value of **R** about ten times the value just mentioned, C becomes constant, and as far as we know remains the same for all higher Reynolds numbers. This phenomenon is very much the same as found by Nikuradse in rough pipes. In fact, it probably has the same cause: the boundary layer is getting thin enough so that the surface roughness affects the flow. Naturally, this occurs sooner in the smaller throats where the relative roughness is greater. There is some evidence that the curve swings a little up and then down again to the constant value, just as the curves for f in Fig. 94 dip down and then up to the constant value.

EXAMPLE

Calculate the flow in gal. per min. in a 4 in. × 2 in. Venturi meter similar to those tested by Mr. Ed S. Smith (a 2-in. throat in a 4-in. pipe), carrying water at 50° F., when the differential head is 3.00 in. of water.

$h = 3.00$ in. $= 0.250$ ft.; therefore, $\sqrt{2\,g\,h} = 8.02 \times 0.5 = 4.01$ ft. per sec. As a first assumption take $V = 0.96$. Then

$V = \dfrac{0.96 \times 4.01}{\sqrt{1 - (1/2)^4}} = \dfrac{3.84 \times 4.01}{\sqrt{15}} = 3.98$ ft. per sec. $\nu = 0.00001410$; therefore, $\mathbf{R} = \dfrac{3.98}{6 \times 0.00001410} = 47{,}000$, and Fig. 108 gives $C = 0.965$. Then

$V = \dfrac{0.965 \times 16.04}{\sqrt{15}} = 4.00$. This will not change **R** enough to change C, and

Q is $\dfrac{4.00\,\pi}{144} = 0.0872$ c.f.s. or $0.0872 \times 448.8 = 39.1$ gal. per min.

PROBLEMS

56–1. A 2-in. × 1-in. Venturi meter similar to those tested by Mr. Ed S. Smith, carries water at 86° F., and the measured differential is 5.88 in. of water. Calculate the flow in gal. per min. *Ans.* 13.7 gal./min.

56–2. What velocity at the throat of the Venturi of the preceding problem would be required to give the same constant, if the temperature of the water were reduced to 4° C.? What velocity of linseed oil at 68° F. would give the same constant? *Ans.* 10.8 ft./sec.; 283 ft./sec.

56-3. Suppose that the Venturi in the preceding problems carries light California crude oil at 68° F., and that the measured differential is 5.88 in. of the oil. What would be the discharge in gal. per min.?

Ans. 12.8 gal./min.

57. Effect of Viscosity on Weirs.—For weirs, the typical linear dimension in the Reynolds number can no longer be the diameter, because a weir has none. Obviously the determining dimension here is the head on the weir. For velocity, there are several possibilities, but the simplest is the ideal velocity corresponding to the head. Therefore, for weirs we will take

$$\mathbf{R} = \frac{H\sqrt{2\,g\,H}}{\nu} = \frac{\sqrt{2\,g}\,H^{3/2}}{\nu} \tag{57.1}$$

For water at 68° F., and measuring the head in feet, this reduces to $\mathbf{R} = 737{,}000\,H^{3/2}$. There have been many careful studies of the flow of water over weirs, but few record the temperature, so that the data are not suited for a study of the variation of the coefficient with the Reynolds number. And studies of the flow of other liquids, which would be more valuable for this purpose than experiments with water, seem to be almost entirely lacking.

However, by carefully studying the data available we can indirectly get some idea of how viscosity affects the coefficient, because we know that this is completely expressed by the way the Reynolds number affects the coefficient: if the viscosity remains constant, the Reynolds number varies as the three-halves power of the head. Experiments on V-notch weirs are especially useful here, because as the head is changed, the shape of the jet remains similar; that is, there is no change in the ratio of H to B, which probably also affects the value of the coefficient for rectangular weirs.

Of experiments on V-notch weirs, those made by James Barr [6] at Glasgow University in 1909 still seem to be the most accurate. The temperature was recorded, but unfortunately the paper as printed gives it for only a few of the runs. Assuming that the temperature of the other runs was about the same, his results for a 90° notch have been reduced to a Reynolds number basis,

[6] "Experiments upon the Flow of Water over Triangular Notches," *Engineering*, Vol. 89 (1910), pp. 435–438, 470–473.

and are plotted in Fig. 109. For the higher Reynolds numbers, the values given by Yarnell[7] have also been incorporated into this curve. Although the observed points scatter from the line a

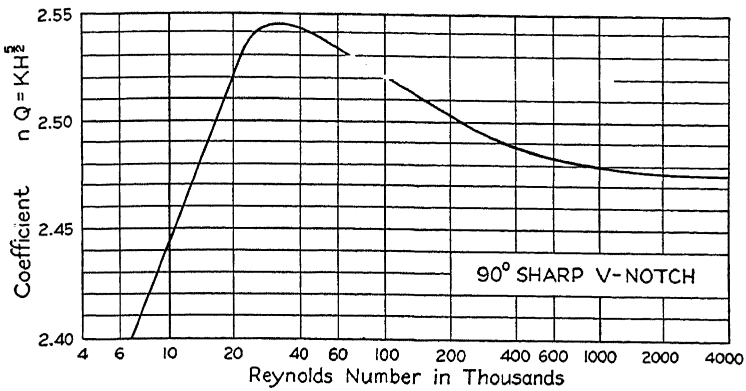


FIG. 109.—Coefficient for 90° sharp V-notch.

few tenths of a per cent, there is little doubt that this curve is fairly close to the truth. The curves for other notch angles are similar, and the resemblance to those for orifices (Fig. 104) is striking. In all of them the coefficient approaches a constant value for high Reynolds numbers. At lower Reynolds numbers it reaches a maximum, and then falls off rapidly for very low Reynolds numbers. The reasons for this have already been discussed in Art. 55. For the 90° notch and Reynolds numbers above 200,000, the curve of Fig. 109 has the empirical equation

$$K = 2.475 + \frac{5500}{\mathbf{R}} \tag{57.2}$$

In connection with Barr's experiments it may be added that he found that the width of the approach channel began to affect the value of the coefficient if it was less than 8 times the head. He found also that the coefficient was affected if the vertex of the notch was less than 4 H above the floor of the approach channel. Since the effect of too narrow a channel was to increase the discharge, and of too shallow a one was to decrease it, a properly proportioned channel will allow the head to exceed the above limits without an appreciable effect on the coefficient.

[7] "Accuracy of the V-Notch Weir Method of Measurement," *Trans. A.S.M.E.*, Vol. 48 (1926), pp. 930–964.

A satisfactory statement as to the effect of variations in viscosity on the discharge over other weirs, such as rectangular notches and full width weirs, must await further study. Experiments by Harris [8] on a full width weir with the head practically constant at 0.178 foot, made at various times of the year so that the temperature varied, showed a quite definite dependence of K on the temperature, its values being 3.425 at 2.2° C. and 3.354 at 25.9° C. His results can be fairly well expressed by the equation

$$K = 3.26 + \frac{5500}{\mathbf{R}}, \tag{57.3}$$

so that the viscosity correction is the same as for V-notch weirs. Unfortunately, the very extensive mass of data on weirs referred to in Note 8 of Art. 25, does not give the temperature. Also, at very low heads surface tension has an effect. The matter is discussed somewhat further in the example and problems.

EXAMPLE

An empirical study of Schoder and Turner's Series D, omitting the runs with "fences," gave the following formula:

$$K = 3.288 + 4.36\left(\frac{H}{D}\right)^2 + \frac{0.0114}{H^{3/2}} - \frac{0.00166}{H^2} \tag{57.4}$$

Compare this with equation (57.3) and the equations given in Probs. 25–11 and 25–15.

As these experiments were made from Dec. 27, 1914, to Jan. 2, 1915, with water drawn directly from Beebe Lake, and the Weather Bureau records at Ithaca show that the air temperature was below freezing, we shall take the viscosity as 0.000019 sq. ft. per sec. Then $\mathbf{R} = \dfrac{8.02\,H^{3/2}}{0.000019} = 422{,}100\,H^{3/2}$, and $\dfrac{0.0114}{H^{3/2}} = \dfrac{4812}{\mathbf{R}}$. This is 12.5 per cent less than $\dfrac{5500}{\mathbf{R}}$, the viscosity term in (57.3). It is over half again as large as the viscosity term in Fteley and Stearns' formula, and nearly three times as great as that in the formula of Prob. 25–11. These two have the surface tension effect combined in one term with the viscosity effect. Equation (57.4) has a separate surface tension term, $-\dfrac{0.00166}{H^2}$. Harris does not give his value of D, but assuming it to be 3 ft.

[8] *Influence of Two Secondary Factors in Weir Measurements*, by C. W. Harris, Bull. 81, Engineering Experiment Station, University of Washington (1935), p. 12.

and taking $H = 0.178$, the velocity of approach term, $4.36\left(\frac{H}{D}\right)^2$, reduces to 0.015; the $-\frac{0.00166}{H^2}$ term becomes -0.052; and $3.288 + 0.015 - 0.052 = 3.251$. This is only 0.3 per cent smaller than the 3.26 of equation (57.3).

Comparing equation (57.4) with those of Probs. 25–11 and 25–15, it is obvious that the constant term is somewhat larger in the latter. This may be due to the omission of a surface tension term. Also, in (57.4) the velocity of approach correction is very much greater. If the runs where "fences" were placed in the channel to make the velocity distribution more uniform had been used, the correction would undoubtedly have been smaller. This must depend on the value of α, and also upon whether the highest velocity is near the surface or not. It is the uncertainty of this term that makes measurement by weirs so unsatisfactory unless they can be calibrated and then used with exactly the same approach conditions. It may be that some day a formula which satisfactorily takes care of all of the factors involved will be devised, but it has not been done as yet notwithstanding the immense amount of labor which has been devoted to the problem.

PROBLEMS

57–1. The highest head used by Schoder and Turner was 2.71 ft. Assuming the temperature of the water to have been 16° C., what was Reynolds number? *Ans.* 2,980,000

57–2. At the head in the preceding problem, would changing the temperature of the water from 32° F. to 86° F. make an appreciable difference in the coefficient? *Ans.* K would change less than 0.002

57–3. With a head of only 0.10 ft., how much would the change of temperature of the preceding problem change the coefficient, if (57.3) is assumed to apply? Does the equation probably apply to a head as low as this?
Ans. 0.233; probably not, since (57.2) did not apply below $\mathbf{R} = 200{,}000$.

57–4. What is the upper limit to values of K in (25.2)?
HINT. C_d in (25.1) can hardly be more than one. *Ans.* 5.35

57–5. What was $\mathbf{R}$ in Prob. 25–9 if the temperature was 20° C.? What is $\mathbf{R}$ for water at 0° C. when $H = 0.40$ ft.? What is $\mathbf{R}$ for water at 58° F. when $H = 0.30$? *Ans.* 183,000; 105,000; 105,000

NOTE. This last problem was suggested by a recent article, "Discharge of V-notch Weirs at Low Heads," by Fred W. Blaisdell in *Civil Engineering*, Aug., 1939, pp. 495–96, which gives $H = 0.3$ ft. as the limit below which variations in head seriously affect K; instead of $H = 0.4$ ft. as given in Art. 25. The problem shows that temperature should not be ignored, 0.4 ft. at freezing being equivalent to 0.3 ft. at 58° F. and to still less at higher temperatures.

58. Effect of Viscosity on Channel Flow.—Published experiments on the flow of viscous liquids in channels seem to be entirely lacking, and until they are available we can only reason from analogy with pipes. For an open channel, Reynolds number may be defined as

(58.1) $$\mathbf{R} = \frac{4\,R\,V}{\nu},$$

where R is the hydraulic radius. This makes the values for semicircular channels and circular channels flowing nearly full the same as that for the corresponding pipe with the same velocity. However, some writers take $\mathbf{R}$ for channels as $\frac{R\,V}{\nu}$. With this latter definition, the transition from laminar to turbulent flow comes at about 500 or 600, while with our definition it would be at about 2000 or 2400.

By solving (42.2) for f and substituting in (50.4), we get

(58.2) $$C = 2.01\,\sqrt{\mathbf{R}},$$

which probably is true for laminar flow in semicircular channels. For channels of other shapes [9] the numerical factor would be different.

When the Reynolds number is high enough so that the flow is turbulent, the shape of the cross-section makes little difference. We may then assume that if the channel bottom and sides are smooth, the flow will be similar to that in a smooth pipe. Solving (42.2) for f, and substituting in (51.2), we get

(58.3) $$C = 32\log_{10}\frac{\mathbf{R}}{C} + 25.7, \text{ or}$$

(58.4) $$\log_{10}\mathbf{R} = \log_{10} C + \frac{C}{32} - 0.804$$

As in the case of the corresponding formula for smooth pipes, the equation cannot be solved for the coefficient in terms of the Reynolds number, but it can readily be solved for the $\mathbf{R}$ corresponding to any C. The curve shown in Fig. 110 was plotted in this manner. Some of Darcy and Bazin's runs on experimental channels lined with neat cement or carefully planed boards are also plotted.

[9] Equation (50.3) applies to laminar flow in circular pipes. For rectangular pipes, S. J. Davis and C. M. White, in *Engineering*, Vol. 78 (1929), p. 71, following J. Boussinesq, *Jour. de Math. pur. et app.* (1868), p. 393; state that for ratios of the two sides of the rectangles of 1 : 1, 1 : 3, 1 : 5, 1 : 10 and 1 : ∞, the 64 in (50.4) should be changed to 57.0, 68.8, 76.2, 85.0, and 96.0, respectively.

The temperature not being given, it was assumed that **R** was 320,000 $R\,V$. (See Prob. 58–2.) It will be seen that C for the

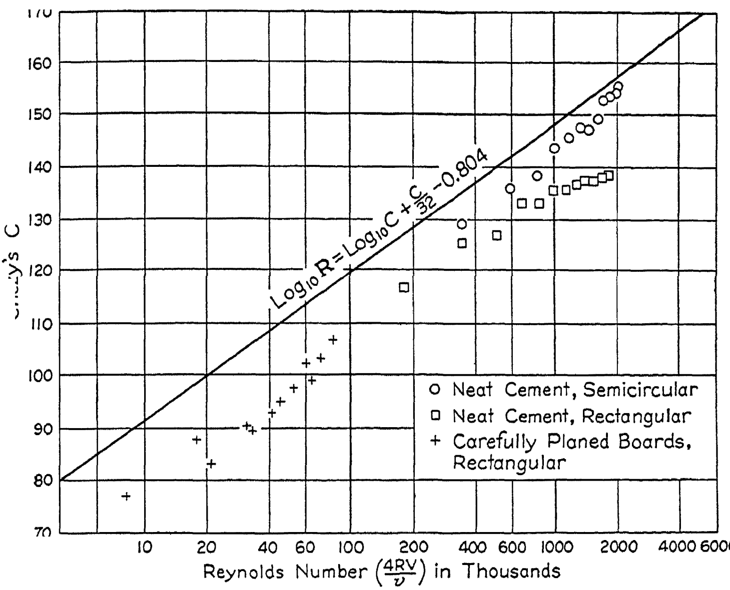


FIG. 110.—Plotting of C against **R** for some of Bazin's experiments.

plotted points averages about 10 per cent less than by (58.4). This corresponds to an f for pipes 19 per cent greater than given by (51.2), which is similar to the pipes mentioned in Art. 53.

EXAMPLE

Find **R** by (58.4) when $C = 160$.

$\frac{C}{32} = 5.000$, $\log_{10} C = 2.204$, and $\log \mathbf{R} = 2.204 + 5.000 - 0.804 = 6.400$.

Therefore $\mathbf{R} = 2{,}510{,}000$.

PROBLEMS

58–1. Find **R** by (58.4) when $C = 80$. *Ans.* 3970

58–2. What temperature of water makes $\mathbf{R} = 320{,}000\,R\,V$? What percentage change in **R** would result from reducing the temperature to 32° F.? From increasing it to 70° F.? *Ans.* 58.1° F.; − 35.3%; 18.0%

58–3. From Fig. 110, estimate the per cent change in C when the temperature is changed from 70° F. to 32° F., when $R = 1$ ft. and $V = 1$ ft. per sec. *Ans.* 5.7%

58-4. Estimate the slope required to make heavy California crude oil at 20° C. flow 1 ft. deep in a rectangular channel 2 ft. wide with a velocity of 5.5 ft. per sec. (Taking this as half a square pipe, $f = \frac{57}{\mathbf{R}}$.)

Ans. 0.0289

58-5. Work the preceding problem for light California crude oil.

Ans. 0.0057

59. The Flow of Gases.—Flow is practically always accompanied by a change in pressure, and changing the pressure of a gas changes its density and generally its temperature, and a change in temperature adds or subtracts heat energy. Therefore if the law of conservation of energy, as expressed by Bernoulli's equation, is to be applied to the flow of gases, it will be necessary in the general case to insert terms to provide for changes in heat content. Also the law of continuity must be written in the form (9.4) instead of (9.5). In fact, the mechanics of gases (or Aerodynamics) is a much more complicated subject than the mechanics of liquids, involving thermodynamics as well as mechanics. For examples of the methods which must be used, the student is referred to Daugherty's *Hydraulics*, 4th Ed., pages 80–82, 177–185, and 252–254.

Where the pressure change does not exceed 10 per cent, problems concerning the flow of gases in pipes and Venturi meters may be worked by treating the gas as a liquid having a uniform density equal to the average density of the gas. Long pipes in which the pressure drop exceeds 10 per cent may be broken up into "reaches." The h_f in the pipe formula will be in feet of this imaginary liquid; that is, in feet of the gas considered as of uniform density. With this understanding, Fig. 94 applies to gases as well as liquids.

EXAMPLE

15,000 cu. ft. per min. of air at 20° C. is to be delivered at atmospheric pressure by a smooth walled rectangular ventilating duct, 3 ft. by 4 ft. in cross-section. Find the pressure drop required per 100 ft. of duct.

15,000 cu. ft. per min. = 250 c.f.s., therefore $V = \frac{250}{12} = 20.83$ ft. per sec. $P_w = 6 + 8 = 14$ ft., and $R = \frac{12}{14} = 0.857$. Then the equivalent diameter of circular pipe is $4 \times 0.857 = 3.428$ ft. ν for air under these conditions is 0.150 stokes. Therefore $\mathbf{R} = \frac{20.83 \times 3.428 \times 929}{0.15} = 442{,}000$. From Fig. 94,

$f = 0.013$. As stated in Art. 53, the formula as plotted in Fig. 94 is for ideally smooth pipes; actual pipes give f five or ten per cent higher. Therefore, we will take $f = 0.014$, and h_f per 100 ft. $= \dfrac{0.014 \times 100 \times 20.83^2}{3.428 \times 64.32}$ $= 2.75$ ft. of air. As stated in Example II of Art. 2, w for air at atmospheric pressure may be taken as 0.0765 lb. per cu. ft.; therefore, the required pressure is $2.75 \times 0.0765 = 0.210$ lb. per sq. ft. $= 0.00146$ lb. per sq. in. As this is only one ten-thousandth of the pressure at the end of the pipe, this method could have been used for a pipe very much longer.

PROBLEMS

59-1. Solve the example if the flow were to be 144,000 cu. ft. per min.
Ans. 0.0964 lb./sq. in./100 ft.

59-2. How much pressure would be required to overcome abrupt entrance and exit loss in the example and in Prob. 59-1?
Ans. 0.0054 lb./sq. in.; 0.496 lb./sq. in.

59-3. Find the pressure required to pump hydrogen at 20° C. and approximately atmospheric pressure through 200 ft. of 1-in. pipe at an average velocity of 20 ft. per sec. Take weight of hydrogen as 0.00520 lb. per cu. ft.
Ans. 0.0234 lb./sq. in.

CHAPTER VIII

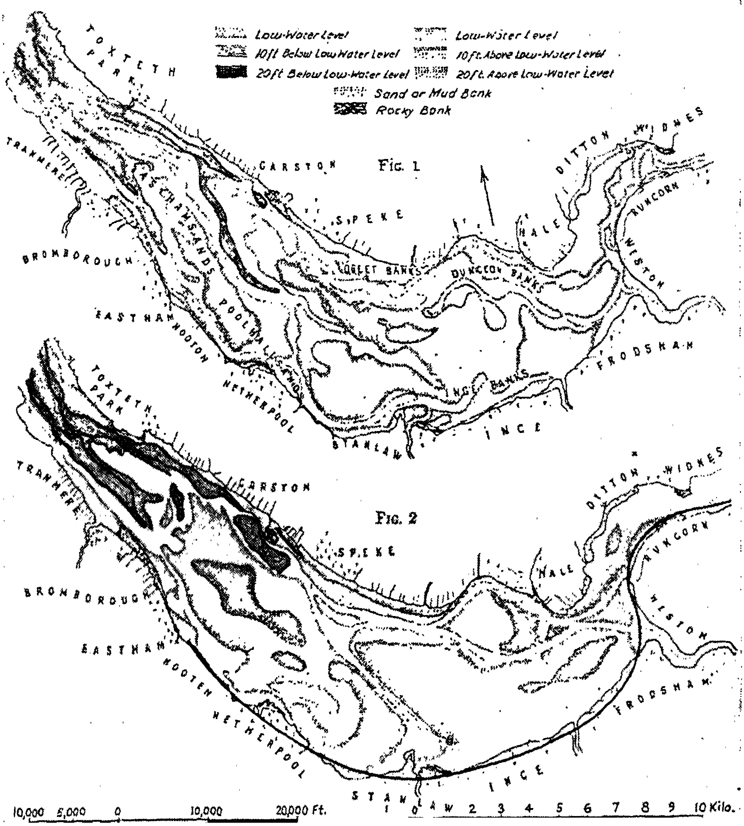
MODELS

60. The Use of Hydraulic Models.—Many problems in the mechanics of liquids are too complicated to be solved by the methods we have been discussing, or they involve constants which cannot be determined in advance of the construction of the structure. For this reason, and also as a further check on the computed design of important structures, the use of reduced scale models has come more and more into use.

Estimates of ship resistance by towing small models are said to have been made in England as early as 1750. Use of ship models came into common use in the 1870's, principally through the work of William Froude. About the middle of the last century, James B. Francis at Lowell, Massachusetts, tested a small model of an inward flow turbine, but probably did not realize all the implications. In 1875 the Frenchman, L. Fargue, made the first model for studying stream flow. The city of Bordeaux, being confronted with the problem of improving the navigation channel of the Garonne, and there being a dispute whether dredging alone would effect a permanent solution, Fargue built a model of the river at one-hundredth full size, put a layer of sand about a foot deep in the bottom, passed a flow of about 7 c.f.s. through it and watched the results. The model convinced the authorities that dredging was not enough, which was probably the true answer; but the vertical scale was not the same as the horizontal, the slope not the same as in nature, and the sand grains much more than one-hundredth the diameter of the sand in the river bed, so that this was not a truly geometric model. What the effect of such discrepancies is, and whether a model truly to scale is necessary or even desirable is discussed in the following articles.

At the same time Osborne Reynolds in England was using models (one 2 feet, 6 inches long, the other 5 feet, 6 inches) to represent screw propelled steamships, and to study the effect of

reversing the screw on the action of the rudder.[1] He advised the Admiralty to construct models of proposed ships at a scale that



Figs. 1 and 2 The Characteristics of Rivers and Estuaries
Fig. 1 Map of the Bed of the Mersey Estuary Fig. 2 Artificially Reproduced Model of Bed

FIG. 111.—Reproduction of figures from Reynolds' paper on models of the Mersey estuary.

would make it possible to install engines and test them as small launches. Later he urged tests on "self righting" life boats by

[1] "On the Steering of Screw Steamers," *Report of British Ass'n*, 1875, or *Scientific Papers*, Vol. I, pp. 134–140.

means of small models,[2] pointing out that opportunities to test the full sized boats in extreme storms were very rare, but that corresponding storms for small models occurred very frequently. "The behavior of a model three feet long in waves two feet high, and with a wind twenty miles an hour, would correspond with that of a boat 27 feet long, in waves 18 feet high, and a velocity of the wind of 60 miles an hour." [2] The ease with which conditions which occur rarely in nature can be produced in a small scale model in the laboratory is one of the great advantages of the use of models. Beginning in 1885 Reynolds made an investigation of certain improvements in the estuary of the Mersey by means of small scale models with sand bottoms.[3] Figure 111 shows the result of one run on a model with a horizontal scale of 1:10,600 and a vertical scale of 1:396 compared with the actual estuary. A pan placed in the water at the downstream end was alternately depressed and raised so as to simulate the action of the tides.

> "The working of the model by hand at once showed that there was only one period of working at which the motion of the water in the model would imitate the motions of the actual tide in the Mersey, which period was found to be about forty [4] seconds; a result that might have been foreseen from the theory of wave motions, since the scale of velocities varies as the square roots of the scale of wave heights, so that the velocities in the model which would correspond to the velocities in the channel would be as the square roots of the vertical scales—about $\sqrt{960}$—and the ratios of the periods would be the ratio of horizontal scales divided by the ratio of velocities, or $\frac{\sqrt{960}}{31,800}$."

These experiments were continued by L. F. Vernon-Harcourt, who also soon gave his attention to the improvement of the mouth

[2] "On Methods of Investigating the Qualities of Lifeboats," *Proc. Manchester Literary and Philosophical Soc.*, Vol. XXVI (1886), or *Scientific Papers*, Vol. II, pp. 321–325.

[3] "On Certain Laws Relating to the Regime of Rivers and Estuaries, and on the Possibility of Experiments on a Small Scale," *Reports of the British Ass'n*, 1887, 1889, 1890, and 1891, or *Scientific Papers*, Vol. II, pp. 380–518.

[4] This refers to a still smaller model, with horizontal scale 1:31,800 and vertical scale 1:960. Then $\frac{\sqrt{960}}{31,800} = \frac{1}{1026}$, and since the tide period in nature was 40,700 sec., in the model it would be $\frac{40,700}{1026} = 39.7$ sec.

of the Seine.[5] These experiments were continued by the French government during the period 1890–1895, and then for some reason not understood by the writer, the use of river models in England and France came to an end.

The scene now shifts to Germany, where Hubert Engels became Professor of Hydraulics for the Technical University of Dresden in 1890, and where in 1898 he opened the first permanent

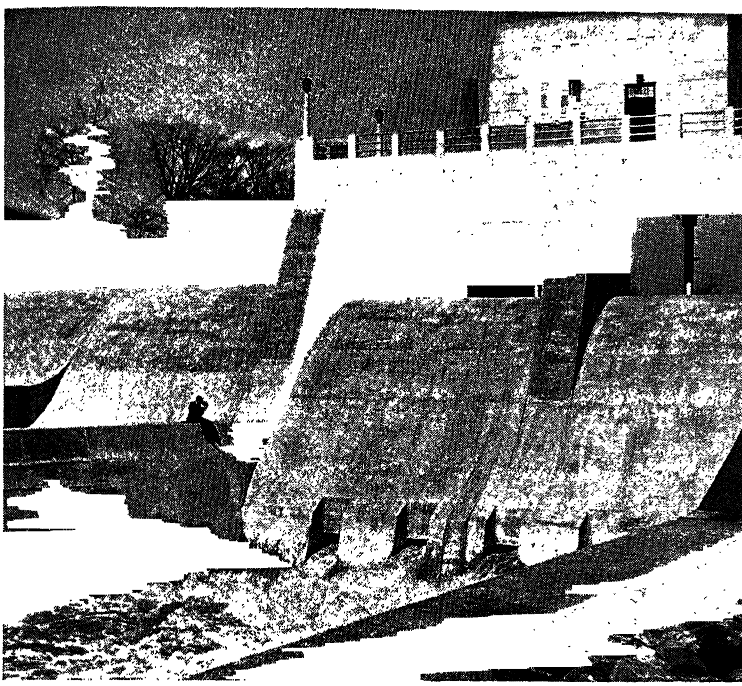


FIG. 112.—Charles Mill dam near Perrysville, Ohio, discharging about 1000 c.f.s. (Courtesy of Professor George E. Barnes.)

river hydraulics laboratory. This was soon followed by similar laboratories at several other European engineering colleges. The underlying theory of hydraulic models was worked out, and skill

[5] "The Principles of Training Rivers through Tidal Estuaries, as Illustrated by Investigation into the Methods of Improving the Navigation Channels of the Estuary of the Seine," *Proc. Royal Soc. of London*, Vol. XLV (1889), pp. 504–524; and "Investigations into the Effects of Training Walls in an Estuary like the Mersey," *Proc. Royal Soc. of London*, Vol. XLVII (1890), pp. 142–144.

in their application to the solution of practical problems developed. In 1924 John R. Freeman visited the laboratories at Berlin, Dresden, Karlsruhe, and Brünn, and was so impressed by what he saw that he arranged for the publication of a book which would collect this new knowledge and make it available to all who wished to use hydraulic models. This was published in German in 1926 and in English in 1929,[6] and has been an immense help in the development of the use of hydraulic models, at least in America. In 1915–16 when the Miami Conservancy

Fig. 113.—1:30 scale model of the dam of Fig. 112. Discharge corresponds to 3590 c.f.s. in prototype. (Courtesy of Professor George E. Barnes.)

District constructed models of the proposed stilling basins for their outlet works,[7] it was a practically unheard of innovation

[6] *Hydraulic Laboratory Practice*, edited by John R. Freeman, A.S.M.E., 1929.

[7] *The Hydraulic Jump as a Means of Dissipating Energy*, by Ross M. Riegel and John C. Beebe, Miami Conservancy District, Technical Reports, Part III (1917).

in this country. Now (1939) hardly any hydraulic structure of importance is constructed without testing it, in whole or in part, by means of models.[8] The U. S. War department has established a large outdoor laboratory for river models at Vicksburg, Mississippi, the Bureau of Standards has a large indoor laboratory at Washington, the Reclamation Bureau has one at Denver, the Tennessee Valley Authority one at Knoxville, and the Soil Conservation Service several. Numerous laboratories with space for model testing are attached to our various universities. The University of Iowa was a pioneer in this work, which has now been developed at many other institutions. The bulletin on "Current Hydraulic Laboratory Research in the United States" (1939), lists 82 model tests being carried on on structures, 11 on streams, and 10 on tidal channels.

61. Hydraulic Similitude.—The structure or natural feature which the model represents is called the *prototype*. The ideal hydraulic model is geometrically similar to its prototype. But geometrical similarity is not enough. There must also be dynamic similarity; that is, velocities in the model must at all points be related to the velocities at the corresponding points in the prototype in some definite way. A little reflection tells us what the relationship must be. The velocity in the model must be in the same direction as the velocity in the prototype, and the magnitude must be such that the velocity heads are in the same ratio as the linear dimensions. For example, if we had water flowing in a glass sided flume 2 feet deep and with a velocity head of 0.4 foot, and if we painted the energy gradient on the glass and photographed the whole arrangement so that our photograph was just one-twelfth size, it would show the water 2 inches deep and the velocity head 0.4 inches. Now if we wished to construct a 1:12 model of this flow, it would have to be exactly like the photograph; that is, the velocity head would have to be 0.4 inches or one-twelfth that in the prototype, and the velocity in the model would equal the velocity in the prototype divided by $\sqrt{12}$.

[8] See, for example, *Model Studies of Spillways*, and *Model Studies of Penstocks and Outlet Works*, Boulder Canyon Project Final Reports, Bull. 1 and 2 (1938), U. S. Bureau of Reclamation; "Construction and Testing of Hydraulic Models, Muskingum Watershed Project," by George E. Barnes and J. G. Jobes, *Trans. A.S.C.E.*, Vol. 103 (1938), pp. 227–255; and "An Ordered Tidal-Canal Flow," by Donald F. Horton, *Eng. News-Record*, June 2, 1938, pp. 785–787.

To put this in mathematical language, let L_r = the ratio of geometrical similarity, V_p = velocity at any point in prototype, and V_m = velocity at the corresponding point in the model. Then the "scale ratio of velocities" $= V_p \div V_m = V_r$.[9] Similarly let the depth at this point in the prototype be D_p and the depth at the corresponding point in the model be D_m; then the "scale ratio of depths" $= D_p \div D_m = D_r = L_r$. Then the requirement for dynamic similarity is [10]

$$L_r = V_r^2 \qquad \text{or} \qquad V_r = L_r^{0.5} \tag{61.1}$$

To have true hydraulic similitude between the model and the prototype, should the surface roughness in the model be the same as in the prototype, or should the relative roughness be the same? Should the same liquid be used in the model as in the prototype, or should the ratio of specific weights depend upon the scale ratio? What about viscosities? These are some of the questions we shall endeavor to answer in the following articles.

It must not be felt that hydraulic similitude is a purely mathematical matter or that absolutely perfect similarity is necessary for practical results. The matter was very well expressed in the words of Vernon-Harcourt regarding his experiments on the Seine.

> "If I succeed in demonstrating with the model that the originally existing conditions can be reproduced typically; and if, moreover, by placing regulating works in the model, the same changes can be reproduced that were brought about by the training works actually built, then I am sure that I can take the third and most important step, namely, of investigating, with every promise of success, the probable effect of the projects that have been proposed for extending the training dikes toward the mouth of the river."

Sometimes a great deal of cut-and-try work is needed in building river models to different vertical and horizontal scales and differ-

[9] "Practical River Laboratory Hydraulics" by Herbert D. Vogel, *Trans. A.S.C.E.*, Vol. 100 (1935), pp. 118–184, uses the reciprocal of this as the scale ratio. That is undoubtedly the logical definition, but the writer agrees with Professor Straub (*ibid.* p. 145) that in computations it is much more convenient to deal with whole numbers rather than fractions.

[10] Sir Isaac Newton, in his *Principia* (1686), Book II, Proposition XXXII, Theorem XXVI, stated that for similar dynamic systems the accelerations must be "as the diameters of the corresponding particles inversely, and the squares of the velocities directly" which is a more general statement of the requirement of (61.1).

ent roughnesses before one can be found that agrees with the prototype for certain known conditions. This process is known as model adjustment and verification.[11] In river work no model should be used to forecast changes until it has been verified with existing conditions. In models of proposed structures, however, this is impossible. The procedure in this case is discussed in the following articles.

EXAMPLE

A model of an Ogee spillway is constructed at a scale of 1:16 across a flume 3 ft. wide. The crest of the spillway is 32 ft. above the bed of the stream. How high is the model crest above the floor of the flume? How much of the spillway does the model represent? What head on the spillway corresponds to 3 in. on the model? At this head, the model discharges 1.50 c.f.s. What discharge per ft. of width in the prototype does this represent?

$$32 \div 16 = 2 \text{ ft.} = \text{height of model crest.}$$
$$3 \times 16 = 48 \text{ ft. of spillway represented.}$$
$$0.25 \times 16 = 4 \text{ ft. of head on prototype.}$$

The velocity in the prototype will be everywhere $\sqrt{16} = 4$ times as great as in the model. The depth over the crest will be 16 times as great and the width 16 times as great, therefore 1.50 c.f.s. in the model corresponds to $4 \times 16 \times 16 \times 1.50 = 1536$ c.f.s. in the prototype. This would be for a width of 48 ft.; therefore, the discharge per ft. of width $= 1536 \div 48 = 32$ c.f.s. per ft. of width. Or, more simply, the model discharges 0.50 c.f.s. per ft. of width, and the prototype which is 16 times as deep and flows 4 times as fast, therefore will discharge $16 \times 4 \times 0.50 = 32$ c.f.s. per ft. of width.

PROBLEMS

61–1. If the mean velocity at exit of the stilling basin below the model in the above example was 1.25 ft. per sec., what would be the corresponding velocity in the prototype? *Ans.* 5.00 ft./sec.

61–2. In the Charles Mill dam shown in Fig. 112, each "bay" of the lower spillway stands 33 ft. above the floor of the outlets, and is 29 ft. wide. What are the corresponding dimensions in the 1:30 model shown in Fig. 113? The gate openings are 3 ft. 6 in. wide by 5 ft. high. What is the area of a corresponding gate opening in the model?

Ans. 1.10 ft.; 0.967 ft.; 0.01944 sq. ft.

61–3. The test shown in Fig. 113 represented a flow of 3590 c.f.s. through 4 gates. What was the velocity through the gates in the model and in the prototype? *Ans.* 9.37 ft./sec.; 51.3 ft./sec.

[11] "The Use and Trustworthiness of Small-Scale Hydraulic Models," by Paul W. Thompson, *Civil Engineering*, Vol. 8 (1938), pp. 255–257.

61-4. Check the statements about a model life boat quoted from Osborne Reynolds in Art. 60.

61-5. Each of the twin tunnels whose downstream portals are shown in Fig. 114 are 20 ft. horseshoe sections with a cross-sectional area of 331.7 sq. ft. When they were discharging a total of 22,000 c.f.s., what was the mean

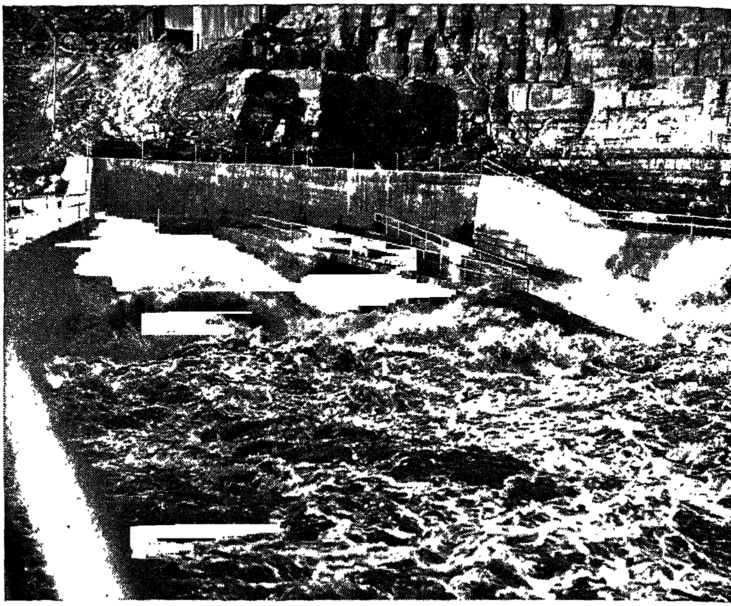


FIG. 114.—Outlet works for Mohawk dam near Warsaw, Ohio. Twin 20 ft. tunnels discharging about 22,000 c.f.s. into stilling basin during Jan. 1937 flood. Water in reservoir about 22 ft. above portal. (Courtesy of Professor George E. Barnes.)

velocity in the tunnels? The scale of the model shown in Fig. 115 was 1:40. When it was discharging a corresponding amount, what was the mean velocity in its tunnels? What was the discharge?

Ans. 33.2 ft./sec.; 5.24 ft./sec.; 2.17 c.f.s.

62. Undistorted Model, Friction Ignored.—There are many problems in which friction effects are so secondary that they may be ignored. Typical of these are the discharge of spillways (whether Ogee, side channel, or morning glory) and the hydraulic jump. To assume that geometrically similar models of these phenomena will follow the law of dynamic similarity stated in (61.1) is an approximation close enough for most practical pur-

poses. If a cross-sectional area in the prototype is A_p and the corresponding area in the model is A_m, $A_p \div A_m = A_r = L_r^2$. The discharge in the prototype will be $Q_p = A_pV_p$, the discharge in the model will be $Q_m = A_mV_m$, and the scale ratio of discharge $= Q_r = Q_p \div Q_m = \dfrac{A_pV_p}{A_mV_m} = A_rV_r = L_r^2 \times L_r^{0.5} = L_r^{2.5}$. This was illustrated in the example of the preceding article and in some of the problems. Notice that in this system of notation any letter

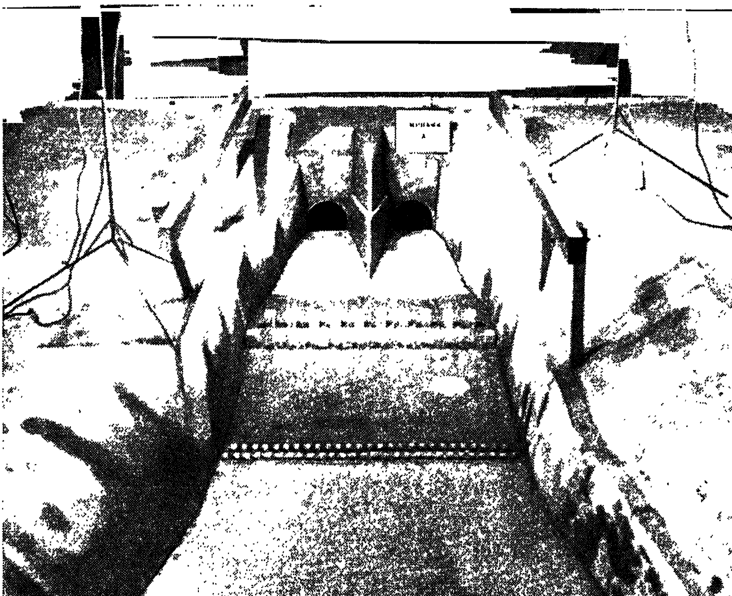


FIG. 115.—1:40 scale model of stilling basin shown in Fig. 114. (Courtesy of Professor George E. Barnes.)

with the subscript p represents that quantity in the prototype, the same letter with the subscript m represents the same quantity in the model, and the same letter with the subscript r indicates the number of times this quantity in the prototype is greater than the corresponding quantity in the model; that is, it is the *scale ratio* for that quantity. It should also be noted that the scale ratio for any desired quantity can always be derived by simply adding the subscript r to all quantities in the defining equation and dropping all constants. That is, since $Q = A\,V$, $Q_r = A_rV_r$.

EXAMPLE I

Derive the scale ratio for the areas of similar circles and for similar velocity heads.

$$A = \pi r^2. \quad \text{Therefore, } A_r = r_r^2 = L_r^2$$

$$h_v = \frac{V^2}{2g}. \quad \text{Therefore, } h_{vr} = V_r^2 g_r^{-1}$$

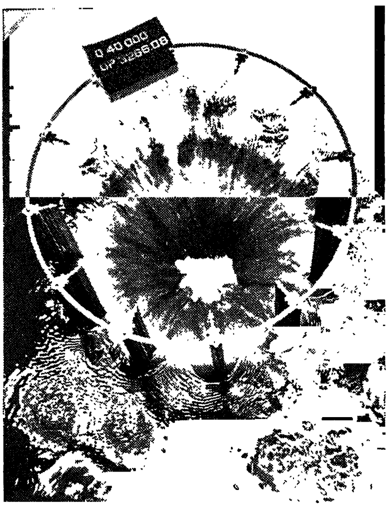


FIG. 116.—1:40 scale model of gate-controlled morning glory spillway of Kingsley dam at Keystone, Nebraska. (Courtesy of Professor George E. Barnes.)

Since actual values of g can vary very little, g_r is practically always unity, and $h_{vr} = L_r = V_r^2$ as given in (61.1). But a more complete form of the equation is $L_r = V_r^2 g_r^{-1}$ or $V_r = L_r^{0.5} g_r^{0.5}$.

Following the method of Example I, and using w_r for the scale ratio of specific weights of the liquids used in prototype and model and g_r for the scale ratio of the acceleration of gravity, Table VI is derived. As pointed out above, g_r is generally unity and can be omitted, but it is retained in the table because it has been suggested that certain kinds of models might be constructed so that they could be rotated rapidly and a resultant acceleration produced which would be many times the ordinary value.

TABLE VI.—SCALE RATIOS, FRICTION IGNORED

QUANTITY	SCALE RATIO
Length, breadth, or height	L_r
Area	$A_r = L_r^2$
Volume	$A_r L_r = L_r^3$
Weight per unit volume	w_r
Total weight	$W_r = w_r L_r^3$
Fluid pressure per unit area	$p_r = \frac{W_r}{A_r} = w_r L_r$ or $p_r \quad w_r h_r = w_r L_r$
Slope	$S_r = \frac{h_r}{L_r} = \frac{L_r}{L_r} = 1$

Quantity	Scale Ratio
Mass	$m_r = \dfrac{W_r}{g_r} = w_r L_r^3 g_r^{-1}$
Velocity	$V_r = L_r^{0.5} g_r^{0.5}$
Discharge	$Q_r = A_r V_r = L_r^{2.5} g_r^{0.5}$
Time	$t_r = \dfrac{L_r}{V_r} = L_r^{0.5} g_r^{-0.5}$
Acceleration	$a_r = \dfrac{V_r}{t_r} = g_r$
Force	$F_r = m_r g_r = w_r L_r^3$
Momentum	$M_r = m_r V_r = w_r L_r^{3.5} g_r^{-0.5}$
Energy or Work	$E_r = m_r V_r^2 = w_r L_r^4$ or $E_r = F_r L_r = w_r L_r^4$
Power	$P_r = \dfrac{E_r}{t_r} = w_r L_r^{3.5} g_r^{0.5}$

EXAMPLE II

A spillway model built to a scale of 1:25 discharges 2 c.f.s. of water with a drop of 2.20 ft. from headwater level to tail water level. To what flow in the prototype does this correspond, and how much power does this fall represent in model and in prototype?

$Q_r = L_r^{2.5} = 25^{2.5} = 3125$, and $Q_p = 2 \times 3125 = 6250$ c.f.s. w_r and g_r are both unity; therefore, $P_r = L_r^{3.5} = 78{,}125$. Power in model $= \dfrac{HQ}{8.8}$ $= \dfrac{2.20 \times 2}{8.8} = 0.50$ hp. Power in prototype $= 0.50 \times 78{,}125 = 39{,}062$ hp. Or, more directly, drop in prototype $= 2.2 \times 25 = 55$ ft. and power $= \dfrac{55 \times 6250}{8.8} = 39{,}062$ hp.

There is, then, a great financial saving in using models of small size, both in the time and money required to build the model, in the pumping capacity and power required to operate it, and in the time required to put the model through any given sequence of operations. The great disadvantage of small scales, however, is that the Reynolds number may be so small as not at all to represent properly the flow in the prototype. This matter will be discussed further in the next article.

PROBLEMS

62–1. Show in detail how the scale ratios in Table VI for volume, discharge, and time were obtained.

62–2. Do the same for mass, force, and power.

62–3. If w_r and g_r are both unity and L_r is 100, what are the scale ratios for velocity, discharge, force, power, and work?

62–4. In the model of Prob. 61–2, find the scale ratio of velocity, discharge, time, and energy. *Ans.* 5.48; 4930; 5.48; 810,000

62–5. A model of a spillway built to a scale of 1 in. = 5 ft. discharges 2.00 c.f.s. What would the full size spillway discharge?

Ans. 55,800 c.f.s.

62–6. A spillway for a maximum discharge of 320,000 c.f.s. is to be tested by an undistorted model in a laboratory where the maximum flow available is 10 c.f.s. What is the largest scale that can be used? *Ans.* 63.4

63. Undistorted Model, Friction Considered.—In river models, or in any problem where frictional resistance cannot be ignored, true similarity requires that Reynolds number in the model be the same as in the prototype. If $\nu_r = \frac{\nu_p}{\nu_m}$ this would require that $\frac{V_r D_r}{\nu_r} = 1$, and, if dynamic similarity is to be preserved, $\nu_r = L_r^{1.50} g_r^{0.50}$. That is, if $g_r = 1$, and $L_r = 25$, $\nu_r = 125$, or the kinematic viscosity of the liquid in the prototype would have to be 125 times that in the model. If the prototype uses water, there is no liquid with a small enough viscosity to use in the model. At ordinary temperatures, the kinematic viscosity of water is about 8 times that of mercury, so that mercury could be used in a 1:4 model, except for the prohibitive cost. If by rapid rotation of the model an acceleration of 80 g could be produced, g_r would be $\frac{1}{80}$. Then for $L_r = 20$, $\nu_r = \frac{20\sqrt{20}}{\sqrt{80}} = 10$, and mercury at a temperature somewhat warmer than the water could be used. But this whole idea seems impracticable.

If friction is the only item to be considered and gravity is not depended on to produce the flow, the attempt to preserve dynamic similarity may be abandoned, and a new sort of similarity developed with a velocity scale $V_r = L_r^{-1}\nu_r$. This would be possible only in closed flow systems (pipes), and there seems little need for model studies in this field. However, the scale ratios for constant Reynolds number are listed in the table at the end of Appendix B.

A method of considering friction effects that has been widely used is to assume that Manning's formula, $V = \frac{1.49}{n} R^{2/3} S^{1/2}$, is correct. Then since S is dimensionless, $V_r = L_r^{2/3} n_r^{-1}$ and a new set of scale ratios based on this can be developed. The first eight will be the same as in Art. 62. The others are given in Table VII.

TABLE VII.—SCALE RATIOS, FRICTION CONSIDERED

QUANTITY	SCALE RATIO
Velocity	$V_r = L_r^{0.67} n_r^{-1}$
Discharge	$Q_r = V_r A_r = L_r^{2.67} n_r^{-1}$
Time	$t_r = L_r \div V_r = L_r^{0.33} n_r$
Acceleration	$a_r = V_r \div t_r = L_r^{0.33} n_r^{-2}$
Force	$F_r = m_r a_r = L_r^{3.33} w_r g_r^{-1} n_r^{-2}$
Momentum	$M_r = m_r V_r = L_r^{3.67} w_r g_r^{-1} n_r^{-1}$
Work and Energy	$E_r = F_r L_r = L_r^{4.33} w_r g_r^{-1} n_r^{-2}$
Power	$P_r = E_r \div t_r = L_r^{4} w_r g_r^{-1} n_r^{-3}$

In actual practice the effort has been made to use a value of n_r that will make these scale ratios the same as in Art. 62; that is, preserve dynamic similarity. Obviously this will be the case when

$$(63.1) \qquad L_r^{2/3} n_r^{-1} = L_r^{1/2} g_r^{1/2} \qquad \text{and} \qquad n_r = L_r^{1/6} g_r^{-1/2}$$

if Manning's formula is correct.

EXAMPLE I

A 1:20 model represents concrete with an n of 0.014. What n should the model have?

Since $g_r = 1$, $n_r = 20^{1/6} = 1.648$, and $n_m = 0.014 \div 1.648 = 0.0085$. There is probably no material as smooth as this, so that on this basis of figuring, if it is necessary to use an undistorted model and represent frictional resistance correctly, a somewhat larger model would be required.

PROBLEMS

63–1. In Example I, if the smoothest material available for the model has an n of 0.010, what would be the scale of the smallest model that would correctly represent friction according to Manning's formula, and preserve dynamic similarity? *Ans.* 1:7.53

63–2. Show in detail how the above scale ratios for time, force, momentum, and power were derived.

63–3. Do the same for discharge, acceleration, and energy, the latter by two methods.

63–4. Solve Prob. 62–3 on the Manning formula assumption, if $n_r = 1$.
Ans. 21.5; 215,000; 4,640,000; 100,000,000; 464,000,000

63–5. Do the same for Prob. 62–4. *Ans.* 9.65; 8690; 3.11; 2,520,000

63–6. Show that if (63.1) is satisfied, the scale ratios of this article for discharge, force, and power, are the same as in Art. 62.

63–7. If n for a certain stream is 0.030, what should be the n for a 1:64 model of it, to satisfy Manning's formula and preserve dynamic similarity? *Ans.* 0.015

63–8. If experiments to find the resistance of a dirigible moving through the air are to be made by measuring the resistance of a model immersed in water. what should be the scale ratio of length and velocities for complete similarity? Is this practicable?

63–9. Show that if Manning's formula is true in the "rough pipe" zone, n must vary as the sixth root of the absolute roughness.

There is another standpoint from which friction in models may be viewed. If the model is a true geometrical model of the prototype, including the surface roughness, the relative roughness of model and prototype will be the same. If in addition, both model and prototype are rough enough, or operate at high enough Reynolds numbers, so that the flow is in the rough pipe zone, or

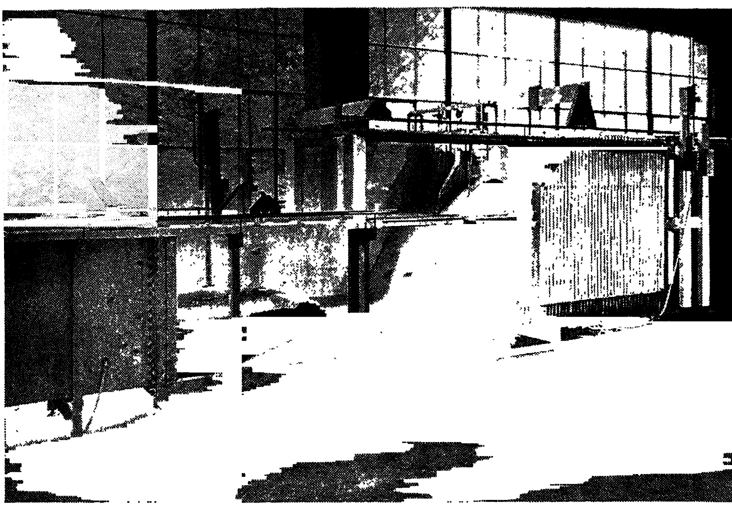


FIG. 117.—1:48 scale model for Mahoning dam near McCrea Furnace Pennsylvania. Note the bank of piezometer tubes at the right. With these the pressure at various points near the dam crest can be measured, and hence the velocity computed. (Courtesy of Professor George E. Barnes.)

the corresponding zone for open channels, f and C will be the same in both model and prototype. Then dynamic similarity will insure the same slope of hydraulic gradient in model and prototype, and true hydraulic similarity will obtain even if the Reynolds number is not the same in the model as in the prototype. In such a situation there would be no advantage in increasing the size of the model to increase its Reynolds number. This is illustrated by the following example.

EXAMPLE II

If the water in Prob. 61–5 (Fig. 114) had a kinematic viscosity of 0.0000125 sq. ft. per sec., and the wetted perimeter of one tunnel was 65.34 ft., what was the Reynolds number? If ϵ was 0.001 ft., what was the relative roughness? By Fig. 94, what f would this give? What is the smallest **R** which would give the same f? A model of what scale ratio would give this **R**?

The hydraulic radius = 331.7 ÷ 65.34 = 5.077 ft. and the equivalent pipe radius may be taken as 10.15 ft. and the equivalent diameter as 20.3 ft. Then $\mathbf{R} = \dfrac{20.3 \times 37.7}{0.0000125} = 61{,}200{,}000$, and $\dfrac{r_0}{\epsilon} = \dfrac{10.15}{0.001} = 10{,}150$. Therefore, from Fig. 94, $f = 0.0105$. From the same figure the smallest **R** giving the same f is 2,000,000. Since $\mathbf{R}_p = \dfrac{V_pD_p}{\nu_p}$ and $\mathbf{R}_m = \dfrac{V_mD_m}{\nu_m}$, for the same viscosity in model and prototype, $\mathbf{R}_r = V_rD_r = L_r^{3/2}$. Then $L_r^{3/2} = 61{,}200{,}000 \div 2{,}000{,}000 = 30.6$ and $L_r = 9.78$.

For this flow a scale of 1:4 would be no better than a scale of 1:9 or 1:9.78. Any model smaller than this will introduce an error, progressively larger the smaller the scale. Making the model any smoother than a relative roughness of 10,150 will not help. For lower heads (and flows) the required model would be larger. However, if the prototype had been rougher, it could have been represented by a smaller model.

For an open channel prototype which falls in the smooth channel range, there seems no practicable way of maintaining com-

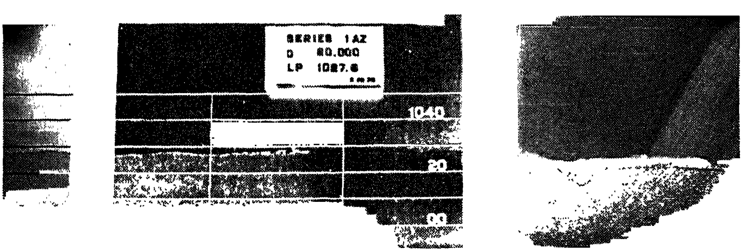


FIG. 118.—Close-up of stilling basin portion of model in Fig. 117, when representing a flow of 20,000 c.f.s. (Courtesy of Professor George E. Barnes.)

plete hydraulic similitude in the model. About the best that can be done is to test a series of models at progressively larger scales (say 1:100, 1:60, and 1:20 as was done in model studies of the Boulder dam spillways). This method has been used extensively in aeronautical model testing. In such testing it is found that if

the prototype value estimated from the several models is plotted against the scale ratio of model, and the curve extrapolated to a scale ratio of 1:1, the result checks fairly well with the prototype. This is called "correcting for scale effect."

PROBLEMS

63–10. A 10 ft. circular tunnel whose ϵ may be taken as 0.00125 ft., carries water at 50 ft. per sec. Taking ν as 0.0000125 sq. ft. per sec., find the scale of the smallest model that would give dynamic similitude and the required ϵ of the model. *Ans.* 15; 0.000083 ft.

63–11. If the n of the tunnel in the preceding problem was 0.012, what n in the 1:15 model would be required if Manning's formula is assumed true? Is there pipe as smooth as this? *Ans.* 0.00764

63–12. By (42.2) and (43.4) find the n corresponding to the prototype data of Prob. 63–10.

64. Distorted Models.—Suppose that it is necessary to construct a model for the data of the Mississippi River given in the example of Art. 42. The scale must not be too small or the flow in the model might become laminar, at least in the shallower portions. As pointed out in the previous article, **R** should be kept fairly large to avoid as much as possible of the "smooth" zone. The Prussian Research Institute for Hydraulics, Marine Design, and Earth Testing, has set a lower limit of $\dfrac{4\,V\,R}{\nu} = 24{,}000$. For ordinary temperatures of water, this leads to the rule that using foot units, $V\,R$ should not be less than 0.075. Then with an undistorted model the maximum value of L_r for the above data would be given by the equations $\dfrac{6.32}{L_r^{0.5}} \times \dfrac{75.91}{L_r} \quad 0.075$, $L_r^{1.5} = \dfrac{6.32 \times 75.91}{0.075}$ and $L_r = 345$. At this scale the wetted perimeter (and therefore the width, approximately) would be 2661 $\div$ 345 = 7.71 feet, and to represent 600 miles of the Mississippi, as was done in one model at Vicksburg,[12] would have required a model length of nearly a mile and three-quarters! To bring such problems within the realm of feasibility, distorted models are used. In fact, as was mentioned in Art. 60, they were used by Fargue and Reynolds in the earliest river models. In the Mississippi

[12] "Observed Effects of Geometric Distortion in Hydraulic Models," by Kenneth D. Nichols, *Proc. A.S.C.E.* (June, 1938), pp. 1081–1102.

River problem just mentioned, the vertical scale was made 1:100 and the horizontal scale 1:2000. For a discussion of the advantages and disadvantages of distortion in models see the paper of Note 12, and also Professor Straub's discussion of Capt. Vogel's paper.[9a]

It is usual in distorted models of rivers to assume that the hydraulic radius is a vertical distance. If the distortion is not too great, little error is introduced by this assumption, as the hydraulic radius of the model will be nearly the same as its mean depth. Making this assumption, writing the vertical scale ratio as D_r, and ignoring friction, we have Table VIII.

TABLE VIII.—SCALE RATIOS, DISTORTED MODELS

QUANTITY	SCALE RATIO
Length and breadth	L_r
Height and depth	D_r
Horizontal area	$A_{hr} = L_r \times L_r = L_r^2$
Vertical area	$A_{vr} = L_r \times D_r = L_rD_r$
Volume	$A_{hr}D_r = L_r^2D_r$ or $A_{vr}L_r = L_r^2D_r$
Weight	$W_r = w_rA_{hr}D_r = w_rL_r^2D_r$
Unit pressure	$p_r = w_rD_r$
Slope	$S_r = D_r \div L_r = L_r^{-1}D_r$
Mass	$m_r = W_r \div g_r = w_rL_r^2D_rg_r^{-1}$
Velocity	$V_r = D_r^{0.5}g_r^{0.5}$
Discharge	$Q_r = V_rA_{vr} = L_rD_r^{1.5}g_r^{0.5}$
Time	$t_r = L_r \div V_r = L_rD_r^{-0.5}g_r^{-0.5}$
Acceleration	$a_r = V_r \div t_r = L_r^{-1}D_rg_r$
Force	$F_r = m_ra_r = w_rL_rD_r^2$
Momentum	$M_r = m_rV_r = w_rL_r^2D_r^{1.5}g_r^{-0.5}$
Work and energy	$E_r = m_rV_r^2 = w_rL_r^2D_r^2$
Power	$P_r = E_r \div t_r = w_rL_rD_r^{2.5}g_r^{0.5}$

If friction is to be considered by assuming Manning's formula to be correct, the first nine of these ratios remain the same, but the last eight become as given in Table IX.

TABLE IX.—SCALE RATIOS, DISTORTED MODELS WITH FRICTION

QUANTITY	SCALE RATIO
Velocity	$V_r = D_r^{2/3}S_r^{1/2}n_r^{-1} = L_r^{-1/2}D_r^{7/6}n_r^{-1}$
Discharge	$Q_r = V_rA_{vr} = L_r^{1/2}D_r^{13/6}n_r^{-1}$
Time	$t_r = L_r \div V_r = L_r^{3/2}D_r^{-7/6}n_r$
Acceleration	$a_r = V_r \div t_r = L_r^{-2}D_r^{7/3}n_r^{-2}$
Force	$F_r = m_ra_r = w_rD_r^{10/3}g_r^{-1}n_r^{-2}$
Momentum	$M_r = m_rV_r = w_rL_r^{3/2}D_r^{13/6}g_r^{-1}n_r^{-1}$
Work and energy	$E_r = m_rV_r^2 = w_rL_rD_r^{10/3}g_r^{-1}n_r^{-2}$
Power	$P_r = E_r \div t_r = w_rL_r^{-1/2}D_r^{9/2}g_r^{-1}n_r^{-}$[illegible]

[9a] See note 9 on page 212.

Usually an attempt is made to adjust n_r so that dynamic similarity obtains; that is, so that $V_r = D_r^{0.5} g_r^{0.5}$. This requires that $D_r^{0.5} g_r^{0.5} = L_r^{-0.5} D_r^{7/6} n_r^{-1}$ or

$$n_r = L_r^{-0.5} D_r^{2/3} g_r^{-0.5} \tag{64.1}$$

By putting n_r and g_r equal to unity this reduces to the so-called "law of compatibility," that n in a distorted model may be kept the same as in the prototype if

$$D_r = L_r^{0.75} \tag{64.2}$$

This law is of little use, however, because with the great variation between size of model and prototype in river models there is much question as to just what is meant in actual roughness by "having the same n," and there is no great point in having them the same anyway. The important matter is that by introducing distortion it is no longer necessary to have the models smoother than they can be conveniently made. The greater the distortion the rougher should be the model.

Sometimes the longitudinal horizontal scale is made different from the transverse; or sometimes the whole model is tilted to change the slope scale independently of the other scales. The corresponding scale ratios can be deduced by the methods used above.

EXAMPLE

The paper of Note 9 gives $Q_r = 1{,}570{,}000$ for a Mississippi River model constructed with a horizontal scale of 1:2400 and a vertical scale of 1:120. Find V_r and n_r.

$A_{vr} = 2400 \times 120 = 288{,}000$; therefore, $V_r = Q_r \div A_{vr} = 1{,}570{,}000 \div 288{,}000 = 5.45$. Then $5.45 = 2400^{-0.5} \times 120^{7/6} \times n_r^{-1}$ and $n_r = \dfrac{120^{7/6}}{5.45\sqrt{2400}} = 0.996$ (by logarithms), or n in the model is 1.004 times n in the prototype.

PROBLEMS

64–1. What would V_r have been in the above example for true dynamic similarity? *Ans.* 10.95

64–2. What kinematic viscosity is assumed in the rule that $V R$ should not be less than 0.075? *Ans.* 0.0000125 sq. ft./sec.

64–3. In another model reported by Capt. Vogel, the horizontal scale was 1:4800, the vertical scale 1:360, and Q_r was 21,000,000. What was n_r? *Ans.* 1.14

64–4. Show how the above scale ratios (neglecting friction) for slope, discharge, acceleration, and power were obtained.

64–5. Do the same for velocity, time, force, and momentum.

64–6. Work the preceding problem if friction is not neglected, but is assumed to follow Manning's formula.

64–7. Work the preceding problem for energy by two methods.

64–8. Check the first nine scale ratios of this article by putting $D_r = L_r$ and comparing with Art. 62.

64–9. Do the same for the last eight scale ratios, comparing with Art. 63.

64–10. Show that another way of expressing the rule of (64.2) is that n will be the same in model and prototype if the distortion ($L_r \div D_r$) equals the cube root of the vertical scale.

64–11. If the tidal period in nature is 40,700 sec., find the period for a tide in Reynolds' model for which the horizontal scale was 1:10,600 and the vertical scale 1:396. *Ans.* 76.4 sec.

64–12. Assuming Manning's formula, derive the scale ratio for velocity and discharge in a distorted model in which the ratio of slope in the prototype to slope in the model is S_r (which may be more or less than $L_r^{-1}D_r$).

Ans. $V_r = D_r^{2/3}S_r^{1/2}n_r^{-1}$; $Q_r = L_rD_r^{5/3}S_r^{1/2}n_r^{-1}$

64–13. The paper referred to in Note 12 gives the following data for a certain Mississippi River model. Horizontal scale 1:450, vertical scale 1:150, slope in prototype 0.00008, slope in model 0.0010, measured discharge scale ratio 317,000. Find n_r. *Ans.* 1.70

64–14. Derive the expression for n_r that will preserve dynamic similarity in the model described in Prob. 64–12, and find the n_r to preserve dynamic similarity in Prob. 64–13. *Ans.* $D_r^{1/6}S_r^{1/2}g_r^{-1/2}$; 0.652

64–15. What model slope in Prob. 64–13 would have preserved dynamic similarity with the roughness used? *Ans.* 0.000147

65. Movable Bed Models.—Many problems which require study by models are of the general nature of those first attacked by Fargue and Reynolds; that is, they have to do with the formation of sand bars in rivers, as to whether erosion will occur below structures, etc. These problems are sometimes studied with models with fixed beds by measuring the velocities near the bed and estimating whether the velocities are small enough to allow deposition or large enough to cause erosion. But this method is unsatisfactory. The usual method is to make the bed of sand or some similar granular material as was done by the pioneers mentioned above. All attempt to preserve geometrical similarity must be abandoned in selecting the size of sand, for the bed of the prototype may be composed of material with a median diameter as small as half a millimeter, and a 1:200 scale ratio would

then require a median diameter in the model of 0.0025 millimeter. Even if a powder of this fineness were available, it is very doubtful if it would work. Instead, a procedure has been worked out based on the following theory.

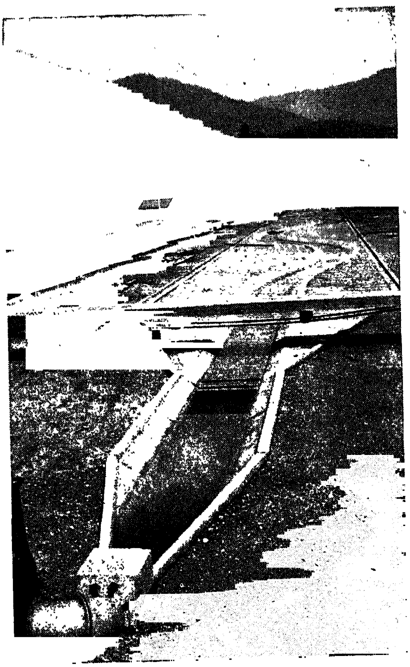


FIG. 119.—Model of a reach of the Yellow River constructed at the Walchensee laboratory in Bavaria. (From "Hydraulic Laboratory Research in Germany" by Herbert D. Vogel, *Civil Engineering*, Nov. 1934, p. 612.)

As was shown in Art. 41, the average shearing stress along the surface of contact of the liquid in a channel and its bed is (for uniform flow)

$$\tau_0 = \frac{w\,S\,A}{P_w} = w\,S\,R \tag{65.1}$$

If the width is large compared with the depth, the mean depth may be substituted for R in the above. In fact, if we take as the free body the liquid above one square foot of bottom where the depth is D feet, and equate the component of the weight parallel to the bottom to the shearing stress on the bottom, we obtain

$$\tau_0 = w\,S\,D \tag{65.2}$$

This is called du Boys' law, having been published in an article, "Le Rhône et les Rivières à Lit Affouillable," by M. P. du Boys, *Annales des Ponts et Chaussées*, Tome XVIII (1879).[13] This value τ_0 is called the *tractive force* per unit area, because if the bed is composed of sand, gravel, or anything movable, this is the force which tends to move it.

Whenever the tractive force reaches a given intensity (for any given sort of sand grain), the smaller grains begin to move.

[13] A complete English translation of this article was prepared by the Bureau of Agricultural Engineering at Iowa City in 1933 under the title, "The Rhone and Streams with Movable Beds."

With an increase in tractive force, the movement becomes more general, and sand riffles form which are miniatures of the sand dunes formed by wind. The sand grains are rolled up the sloping upstream side of the riffle and fall motionless at the steep downstream side, and the riffle progressively moves downstream. In some streams the material moved is much coarser, ranging through various sizes of gravel up to large boulders. The general name for it is *geschiebe* (German for "that which is shoved").[14]

Some particular degree of movement (generally small) is selected as a standard, and the tractive force that will cause it is called the *critical tractive force.* As it is difficult to observe conditions at the bottom of most rivers, a large sample of the geschiebe from the stream is transported to the laboratory and formed into a level bottom in a tilting flume. Water is then caused to flow over it at progressively greater depths or slopes until the critical movement occurs, care being taken that the flow is so adjusted by tail-gate and head-gate that the water surface will be parallel to the tilted bottom. Similarly, the material that is to be used in the model is tested and its critical tractive force determined. This should be less than that in the prototype, and since it is generally not practical to reduce the size very much, recourse is had to using lighter materials such as powdered coal, rosin, etc. (See the reference in Note 12.) Then from a consideration of the model scale ratios, the slope of the model necessary to similate bed movement may be computed. The method is shown by the following example taken from Vogel's paper (Note 9).

EXAMPLE

A model to study bed movement is to be constructed with a horizontal scale of 1:500 and a vertical scale of 1:125. A large sample of bed-load from the prototype, tested at a slope of 0.002, begins to have general movement at a depth of flow of 0.50 ft. The sand to be used in the model test has, for a similar general movement, a critical tractive force of 0.008 lb. per sq. ft. What slope should the model have if the slope in nature is 0.0001?

By (65.2) the critical tractive force of the geschiebe of the prototype is $62.5 \times 0.002 \times 0.50 = 0.0625$ lb. per sq. ft. Then in nature the depth at

[14] For further information on this whole subject see "Sand Mixtures and Sand Movement in Fluvial Models," by Capt. Hans Kramer, *Trans. A.S.C.E.*, Vol. 100 (1935), pp. 798–878, with discussions.

which general movement begins would be $\frac{\tau_0}{wS} = \frac{0.0625}{62.5 \times 0.0001} = 10$ ft. (This could, of course, have been obtained by proportion without introducing the 62.5.) Then since the vertical scale is 1:125, general movement must begin in the model at a flow depth of $\frac{10}{125} = 0.08$ ft. The slope required in the model, therefore, is $S = \frac{\tau_0}{wD} = \frac{0.008}{62.5 \times 0.08} = 0.0016$. The slope resulting from the vertical distortion is only 0.0004, so that an additional slope distortion must be introduced.

The above result could have been obtained in another way. The scale ratio of tractive forces resulting from (65.2) is

$$\tau_{0r} = \frac{\tau_{0p}}{\tau_{0m}} = w_r S_r D_r \tag{65.3}$$

Since water is being used in both model and prototype, w_r is unity. $\tau_{0r} = \frac{0.0625}{0.008} = 7.81$, $S_r = \frac{\tau_{0r}}{D_r} = \frac{7.81}{125} = 0.0625$. That is, the slope in the model must be 16 times the slope in the prototype, as was found above.

Lieut. Nichols, in the paper referred to in Note 12, states that "experience indicates that a combination of moderate depth distortion, moderate slope distortion, and light-weight bed material is better than the use of one extreme to the exclusion of the others."

PROBLEMS

65-1. Draw a clear figure and derive (65.2) from first principles.

65-2. What is the tractive force on the bed of a river at a point where the depth is 10 ft. and the slope 0.0004? *Ans.* 0.25 lb./sq. ft.

65-3. What horizontal scale in the example would have produced the required slope without further slope distortion? *Ans.* 1:2000

65-4. Suppose that in the example, powdered rosin with a critical tractive force of 0.002 lb. per sq. ft. had been used for the bed of the model. What slope should then be used if the other data remain as given in the example? *Ans.* 0.000400

65-5. Rework the example with all the data as there given except that the vertical scale is changed to 1:200. *Ans.* 0.00256

65-6. Rework the preceding problem with 1:100 instead of 1:200. *Ans.* 0.00128

65-7. Keeping the other data the same as given in the example, what vertical scale would have given the proper slope scale without further distortion? *Ans.* 1:62.5

APPENDIX A

PROPERTIES OF LIQUIDS, WITH TABLES

66. Properties of Liquids.—The fundamental property of a fluid as distinguished from a solid is that while a solid, no matter how plastic, requires a certain minimum shearing stress before it will flow; a fluid, no matter how viscous, will flow under action of shearing stress no matter how small, the time rate of deformation at low Reynolds numbers being proportional to the stress. Fluids are divided into liquids and gases. A gas will expand and fill any container in which it is placed; but a liquid, if placed in a container larger than necessary, forms a free surface. Roughly, we may say that liquids have constant volume, or are incompressible, while gases are compressible. However, as pointed out below, liquids are also compressible, though to a much smaller degree than gases.

Probably the physical property of a liquid that enters into more sorts of problems than any other is its specific weight, or weight per unit volume. This varies with the temperature, generally being less at higher temperatures, although water at temperatures just above freezing presents an exception. The specific weight of distilled water free from air and at atmospheric pressure is given in Table X. The actual water encountered in engineering work contains impurities in solution (and often in suspension) which tend to increase its weight. However, it also contains more or less air, which makes it lighter; so that the two effects may counterbalance. If it is necessary to work to four significant figures, the specific weight must be found by careful measurements. For approximate work, 62.5 is a very convenient figure to use for tap water at low temperatures, as it is one-sixteenth of 1000 (i. e., the reciprocal of 62.5 is 0.016—a convenient value to use in computations). The specific weight of any other liquid is the specific weight of water at maximum density (62.428 pounds per cubic foot) multiplied by the specific gravity of the liquid. The latter is numerically the same as its density in the c.g.s. system.

All liquids are somewhat compressible, and the compressibility is generally given quantitatively in the form of its reciprocal, the bulk modulus of elasticity. The latter is defined by the equation

$$E_v = \frac{-\,dp}{\dfrac{dv}{v}} = -\,v\frac{dp}{dv} \tag{66.1}$$

That is, it is the limit approached by the decrease in pressure divided by the resulting relative increase in volume as the change approaches zero. The ratio $\frac{dv}{v}$ is dimensionless, so that E_v is in the same units as dp. The values for water at atmospheric pressure are given in Table X.[1] The values at higher pressures are about 0.028 per cent higher for each atmosphere increase in pressure, but in most problems the change may be neglected.[2]

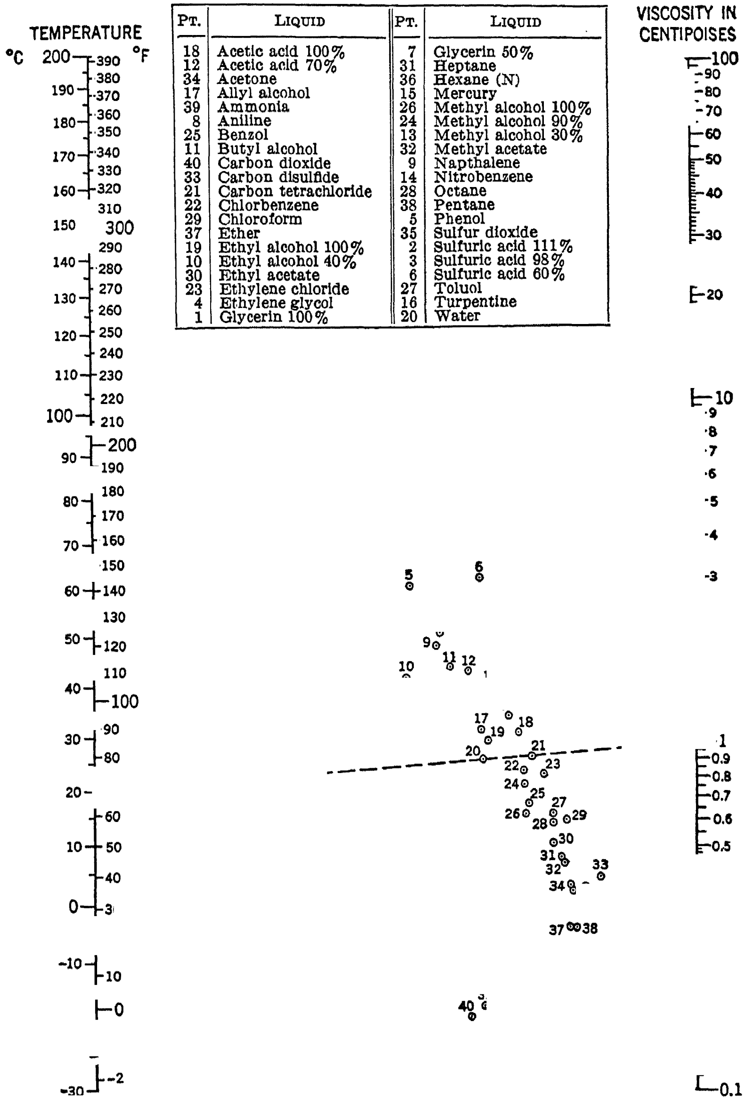
Table X.—Specific Weight and Compressibility of Water

Temperature in Degrees F.	Weight Lb. per Cu. Ft.	E_v Lb. per Sq. In.
32	62.42	292,000
39.2	62.428	299,000
50	62.41	308,000
60	62.37	315,000
68	62.32	320,000
80	62.22	325,000
90	62.12	328,000
100	62.00	330,000
120	61.72	332,000
150	61.20	328,000
200	60.13	308,000
212	59.84	302,000

Another very important property of liquids is viscosity. The viscosity of some substances was listed in Table IV of Art. 48. Figure 120 is an alignment chart giving the viscosity in centipoises

[1] These are based on the average of four sets of somewhat conflicting data. See reference in Note 2.

[2] For further information see "Some Physical Properties of Water and Other Fluids," by Robert L. Daugherty, *Trans. A.S.M.E.* (July, 1935), pp. 193–196. Also his *Hydraulics*, 4th Ed.



PT.	LIQUID	PT.	LIQUID
18	Acetic acid 100%	7	Glycerin 50%
12	Acetic acid 70%	31	Heptane
34	Acetone	36	Hexane (N)
17	Allyl alcohol	15	Mercury
39	Ammonia	26	Methyl alcohol 100%
8	Aniline	24	Methyl alcohol 90%
25	Benzol	13	Methyl alcohol 30%
11	Butyl alcohol	32	Methyl acetate
40	Carbon dioxide	9	Napthalene
33	Carbon disulfide	14	Nitrobenzene
21	Carbon tetrachloride	28	Octane
22	Chlorbenzene	38	Pentane
29	Chloroform	5	Phenol
37	Ether	35	Sulfur dioxide
19	Ethyl alcohol 100%	2	Sulfuric acid 111%
10	Ethyl alcohol 40%	3	Sulfuric acid 98%
30	Ethyl acetate	6	Sulfuric acid 60%
23	Ethylene chloride	27	Toluol
4	Ethylene glycol	16	Turpentine
1	Glycerin 100%	20	Water

FIG. 120.—Alignment chart giving viscosity of various liquids. (From "Viscosity Data in Graphical Form," by Raymond P. Genereaux, *Industrial and Engineering Chemistry*, Vol. 22 (1930), p. 1384.)

of 40 liquids at various temperatures, taken from the paper "Viscosity Data in Graphical Form" by Raymond P. Genereaux, *Industrial and Engineering Chemistry*, Vol. 22 (1930), pp. 1382–1385. To use it, find the number of the liquid in the list at the top and lay a straight edge through the corresponding point and the temperature. The intersection of this line with the other line gives the viscosity. The dashed line illustrates the method for water at 20° C. Values of the absolute and kinematic viscosities of pure water at various temperatures are listed in Table XI. Values for higher temperatures are given in *Fluidity and Plasticity* by Eugene C. Bingham, McGraw-Hill Book Co. (1922), pp. 339–340.

Other properties which sometimes enter into problems in the mechanics of liquids are surface tension and vapor pressure. For values of these, and of the specific gravity, compressibility, and viscosity of other liquids at other temperatures, see *Handbook of Chemistry and Physics*, Chemical Rubber Publishing Co., Cleveland, Ohio, or *International Critical Tables*, McGraw-Hill Book Co. (1933).

67. Miscellaneous Tables.—Tables in the body of the text to which the student may wish to make frequent reference are:

A few other tables which will be found useful are given below. The table of conversion equivalents is condensed from that given in *Hydraulic and Excavation Tables*, published by the U. S. Bureau of Reclamation, and is presented here with their permission. A convenient table of velocity heads corresponding to various velocities is given in *Handbook of Hydraulics* by H. W. King, McGraw-Hill Book Co. (1929), pp. 66–68.

TABLE XI.—VISCOSITY OF PURE WATER

Temperature		Absolute Viscosity C.G.S.	Kinematic Viscosity	
C.°	F.°		Sq. Cm./Sec.	Sq. Ft./Sec.
0	32.0	0.01794	0.01794	0.00001931
2	35.6	.01674	.01674	1802
4	39.2	.01568	.01568	1687
6	42.8	.01473	.01473	1585
8	46.4	.01387	.01387	1493
10	50.0	0.01310	0.01310	0.00001410
12	53.6	.01239	.01240	1334
14	57.2	.01175	.01176	1265
16	60.8	.01116	.01117	1202
18	64.4	.01060	.01062	1143
20	68.0	0.01009	0.01010	0.00001088
22	71.6	.00961	.00963	1036
24	75.2	.00916	.00919	988
26	78.8	.00875	.00877	944
28	82.4	.00836	.00839	903
30	86.0	0.00800	0.00804	0.00000865
32	89.6	.00767	.00771	830
34	93.2	.00736	.00740	796
36	96.8	.00706	.00711	765
38	100.4	.00679	.00684	736
40	104.0	0.00654	0.00659	0.00000709
42	107.6	630	635	684
44	111.2	608	613	660
46	114.8	587	593	638
48	118.4	568	574	618
50	122.0	0.00549	0.00556	0.00000598
52	125.6	532	539	580
54	129.2	515	523	562
56	132.8	499	507	546
58	136.4	484	492	530

To reduce centipoises to slugs per ft.-sec., or to sec.-lb. per sq. ft., multiply by 0.0000209. To reduce centipoises to lb. (of mass) per ft.-sec., or to sec.-poundals per sq. ft., multiply by 0.000672. To reduce stokes (sq. cm. per sec.) to sq. ft. per sec., divide by 929.

TABLE XII.—CONVENIENT EQUIVALENTS

1 inch = 0.027778 yard = 0.00015783 mile = 2.54 centimeters.
1 foot = 0.00018939 mile = 0.3048 meter.
1 mile = 63,360 inches = 5280 feet = 1760 yards = 1.60935 kilometers.
1 meter = 39.37 inches = 3.2808 feet = 1.0936 yards.

1 square inch = 0.006944 square foot = 6.45163 square centimeters.
1 acre = 43,560 square feet = 0.0015625 square mile.
1 square mile = 27,878,400 square feet = 640 acres.
1 square meter = 1550 square inches = 10.7639 square feet.

1 cubic inch = 0.004329 U. S. gallon = 16.3872 cubic centimeters.
1 U. S. gallon = 231 cubic inches = 0.13368 cubic foot = 3.78543 liters.
1 cubic foot = 1728 cubic inches = 7.4805 U. S. gallons = 28.317 liters.
1 cubic centimeter = 0.061023 cubic inch = 0.00026417 U. S. gallon.

1 c.f.s. = 448.8 U. S. gallons per minute = 26,930 U. S. gallons per hour.
1 c.f.s. = 0.9917 acre-inch per hour = 1.9835 acre-feet per day.
1 million gallons per day = 1.55 c.f.s. = 3.07 acre-feet per day.
1 c.f.s. for 1 year will cover 1 square mile 1.131 feet deep.
1 inch deep on 1 square mile = 53.33 acre-feet = 0.0737 c.f.s. for 1 year.

1 avoirdupois pound = 7000 grains = 0.4536 kilogram.
1 foot per second = 0.68182 mile per hour = 1.0973 kilometers per hour.
1 kilogram = 1000 grams = 2.2046 pounds avoirdupois.
1 horsepower = 550 foot-pounds per second = 746 watts.
1 B.t.u. = 778 foot-pounds.

TABLE XIII.—STANDARD STEEL PIPE SIZES

Diameter (In.)		Internal Area		Diameter (In.)		Internal Area	
Nominal	Actual Internal	Sq. In.	Sq. Ft.	Nominal	Actual Internal	Sq. In.	Sq. Ft.
1/8	0.269	0.0568	0.00039	3	3.068	7.39	0.0513
1/4	0.364	0.1040	0.00072	3 1/2	3.548	9.89	0.0687
3/8	0.493	0.1908	0.00132	4	4.026	12.73	0.0884
1/2	0.622	0.304	0.00211	5	5.047	20.01	0.1389
3/4	0.824	0.533	0.00370	6	6.065	28.89	0.2006
1	1.049	0.864	0.00600	7	7.023	38.74	0.2690
1 1/4	1.380	1.496	0.01039	8	7.981	50.03	0.3474
1 1/2	1.610	2.036	0.01414	9	8.937	62.79	0.4360
2	2.067	3.355	0.02330	10	10.020	78.85	0.5476
2 1/2	2.469	4.788	0.03325	12	12.000	113.10	0.7854

TABLE XIV.—AREAS OF CIRCLES

DIAMETER (In.)	AREA (Sq. In.)	AREA (Sq. Ft.)
1	0.7854	0.00545
2	3.142	0.0218
3	7.069	0.0491
4	12.566	0.0873
5	19.635	0.1364
6	28.27	0.196
7	38.48	0.267
8	50.27	0.349
9	63.62	0.442
10	78.54	0.545
12	113.1	0.785
14	153.9	1.069
16	201.1	1.396
18	254.5	1.767
20	314.2	2.18
24	452.4	3.14
28	615.8	4.28
32	804.2	5.58
36	1017.9	7.07
40	1256.6	8.73
44	1520.5	10.56
48	1809.6	12.57
52	2124	14.75
56	2463	17.10
60	2827	19.63
66	3421	23.8
72	4072	28.3
78	4778	33.2
84	5542	38.5
90	6362	44.2
96	7238	50.3
102	8171	56.7
108	9161	63.6
114	10207	70.9
120	11310	78.5

TABLE XV.—VELOCITY HEAD (FEET)

V	0.0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
0	.0000	.0002	.0006	.0014	.0025	.0039	.0056	.0076	.0100	.0126
1	.0155	.0188	.0224	.0263	.0305	.0350	.0398	.0449	.0504	.0561
2	.0622	.0686	.0752	.0822	.0895	.0972	.1051	.1133	.1219	.1308
3	.1399	.1494	.1592	.1693	.1797	.1905	.2015	.2128	.2245	.2365
4	.2488	.2613	.2743	.2875	.3010	.3148	.3290	.3434	.3582	.3733
5	0.389	0.404	0.420	0.437	0.453	0.470	0.488	0.505	0.523	0.541
6	0.560	0.579	0.598	0.617	0.637	0.657	0.677	0.698	0.719	0.740
7	0.762	0.784	0.806	0.829	0.851	0.874	0.898	0.922	0.946	0.970
8	0.995	1.020	1.045	1.071	1.097	1.123	1.150	1.177	1.204	1.231
9	1.259	1.287	1.316	1.345	1.374	1.403	1.433	1.463	1.493	1.524
10	1.555	1.586	1.618	1.649	1.682	1.714	1.747	1.780	1.813	1.847
11	1.881	1.916	1.950	1.985	2.021	2.056	2.092	2.128	2.165	2.202
12	2.239	2.276	2.314	2.352	2.391	2.429	2.468	2.508	2.547	2.587
13	2.627	2.668	2.709	2.750	2.792	2.833	2.876	2.918	2.961	3.004
14	3.047	3.091	3.135	3.179	3.224	3.269	3.314	3.360	3.405	3.452
15	3.498	3.545	3.592	3.639	3.687	3.735	3.784	3.832	3.881	3.930
16	3.980	4.030	4.080	4.131	4.182	4.233	4.284	4.336	4.388	4.440
17	4.493	4.546	4.599	4.653	4.707	4.761	4.816	4.871	4.926	4.981
18	5.037	5.093	5.150	5.207	5.264	5.321	5.379	5.437	5.495	5.554
19	5.613	5.672	5.732	5.791	5.851	5.912	5.973	6.034	6.095	6.157
20	6.219	6.281	6.344	6.407	6.470	6.534	6.598	6.662	6.726	6.791
21	6.856	6.922	6.988	7.054	7.120	7.187	7.254	7.321	7.389	7.457
22	7.525	7.593	7.662	7.731	7.801	7.871	7.941	8.011	8.082	8.153
23	8.224	8.296	8.368	8.440	8.513	8.586	8.659	8.733	8.807	8.881
24	8.955	9.030	9.105	9.181	9.256	9.332	9.409	9.485	9.562	9.639
25	9.72	9.79	9.87	9.95	10.03	10.11	10.19	10.27	10.35	10.43
26	10.51	10.59	10.67	10.75	10.84	10.92	11.00	11.08	11.17	11.25
27	11.33	11.42	11.50	11.59	11.67	11.76	11.84	11.93	12.02	12.10
28	12.19	12.28	12.36	12.45	12.54	12.63	12.72	12.81	12.90	12.99
29	13.08	13.17	13.26	13.35	13.44	13.53	13.62	13.71	13.81	13.[illegible]

From the formula $V = 8.02\sqrt{H}$. If the decimal point is moved in the velocity, it must be moved twice as many places in the head. For more extensive tables, see *Handbook of Hydraulics* by H. W. King, McGraw-Hill Book Co. (1929), pp. 63–68.

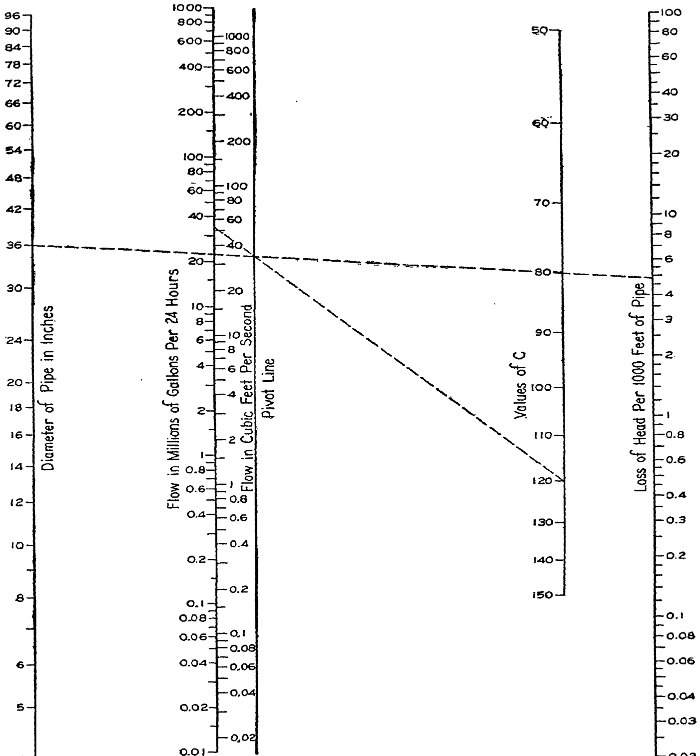


FIG. 121.—Alignment chart for solving Hazen-Williams formula.

APPENDIX B

DIMENSIONAL ANALYSIS

68. Dimensions.—During the last few years, much use has been made in fluid mechanics (and other subjects as well) of a method called *dimensional analysis*. Before presenting it, it will be necessary to say something about dimensions in general. The word is used here with a somewhat different meaning than when we speak of putting the dimensions on a drawing, or when in mathematics we speak of working in two, three, or four dimensions. Here we mean the sort of quantity measured. For example a distance, whether measured in inches, feet, miles, or meters, always has the dimension of length; but time cannot be measured in these units and is essentially a different dimension. Quantities are of two kinds, dimensional, and dimensionless. The latter are ratios or pure numbers. Thus, in the formula $C = 2\pi r$, C and r are dimensional (in this case length), and 2 and π are dimensionless.

All of the dimensional quantities used in mechanics can be expressed in terms of three fundamental units, the others being derived from them. Thus, velocity may be measured in feet per minute, miles per hour, centimeters per second, etc., but will always be a unit of length divided by a unit of time. This is expressed by saying that velocity has the dimensions of length divided by time and is written $[V] = [L\,t^{-1}]$. Acceleration has the dimensions of velocity divided by time and therefore $[a] = [L\,t^{-2}]$. The three units most commonly used as the fundamental ones are length, time, and mass. The others are then called derived units and are expressed in terms of these three. However length, time, and force are sometimes taken as the fundamental units; and other combinations of three might be used. For example, velocity might be taken as a fundamental unit and time as a derived unit of the dimensions $[t] = [L\,V^{-1}]$. For some purposes length, time, force, and mass may all be considered as fundamental, but they are definitely bound together by the fact of nature which was put into words by Newton, and which in our notation becomes

(68.1) $$[F] = [m\,L\,t^{-2}]$$

It will be seen that these "dimensional" equations are written much as were the scale ratios of Chapter VIII, by omitting all dimensionless quantities from the defining equations, and enclosing the residue in square brackets. These brackets indicate that the equation no longer holds for numerical values, but simply indicates the dimensions, that is the essential nature, of the quantities involved. The following table lists the dimensions of all the quantities ordinarily used in mechanics, on the basis of both the m − L − t and the F − L − t systems of fundamental units.

TABLE XVI.—DIMENSIONS OF PHYSICAL QUANTITIES

QUANTITY	SYSTEM (m − L − t)	(F − L − t)
Length	$[L]$	$[L]$
Area	$[L^2]$	$[L^2]$
Volume	$[L^3]$	$[L^3]$
Velocity	$[L\,t^{-1}]$	$[L\,t^{-1}]$
Acceleration	$[L\,t^{-2}]$	$[L\,t^{-2}]$
Angular velocity	$[t^{-1}]$	$[t^{-1}]$
Angular acceleration	$[t^{-2}]$	$[t^{-2}]$
Mass	$[m]$	$[F\,L^{-1}t^2]$
Force and weight	$[m\,L\,t^{-2}]$	$[F]$
Specific weight	$[m\,L^{-2}\,t^{-2}]$	$[F\,L^{-3}]$
Pressure and unit stress	$[m\,L^{-1}\,t^{-2}]$	$[F\,L^{-2}]$
Impulse and momentum	$[m\,L\,t^{-1}]$	$[F\,t]$
Discharge, volume per second	$[L^3\,t^{-1}]$	$[L^3\,t^{-1}]$
Discharge, weight per second	$[m\,L\,t^{-3}]$	$[F\,t^{-1}]$
Energy and work	$[m\,L^2\,t^{-2}]$	$[F\,L]$
Power	$[m\,L^2\,t^{-3}]$	$[F\,L\,t^{-1}]$
Torque	$[m\,L^2\,t^{-2}]$	$[F\,L]$
Density	$[m\,L^{-3}]$	$[F\,L^{-4}\,t^2]$
Absolute viscosity	$[m\,L^{-1}\,t^{-1}]$	$[F\,L^{-2}\,t]$
Kinematic viscosity	$[L^2\,t^{-1}]$	$[L^2\,t^{-1}]$
Surface tension	$[m\,t^{-2}]$	$[F\,L^{-1}]$
Modulus of elasticity	$[m\,L^{-1}\,t^{-2}]$	$[F\,L^{-2}]$

It should be noted that two quantities which have the same dimensions are not necessarily the same. Thus torque is quite different from energy; and in strength of materials, section modulus has the dimension $[L^3]$, but it is not a volume. On the other hand, work and energy are essentially the same, and modulus of elasticity is a unit stress, which is the same as a unit pressure.

One use of the above ideas is in converting from one system of units to another, as illustrated in Example I. Another is in checking mathematical derivations for errors. An equation which states in the simplest form [1] the relationship between physical quantities must be *dimensionally homogeneous;* that is, each term of the equation must be of the same dimensions. If the equation is dimensionally homogeneous, that does not prove that it is correct, but if we start with equations which are homogeneous, and deduce one which is not, we can be sure that we have made a mistake, or that we have added things that should not be added, as in Note 1. Example II illustrates this use of dimensions.

Another use of the principle of dimensional homogeneity is in weighing the value of formulas which have been derived empirically. For example, Kutter's formula (43.1) has three terms in the numerator, two of which are dimensionless (since S is dimensionless). Then if the numerator is to be homogeneous, n must be dimensionless. But since R has the dimension of length, to make the denominator homogeneous would require $[n] = [L^{0.5}]$. Thus it is impossible to make the whole equation homogeneous, and whatever the value of Kutter's formula for obtaining numerical results in the range of the data from which it was derived, it is quite certain that it is not the ultimate statement of the relationship between the quantities involved. On the other hand, the mere fact that an empirical formula is homogeneous does not make it satisfactory. It simply removes one objection and leaves the matter open to be decided on other grounds.

EXAMPLE I

Use the above principles to compute the conversion factor for finding the viscosity of water at 50°F. in the slug-ft.-sec. system, having given that it is 0.01310 in the c.g.s. system.

Since the dimensions of viscosity are $[m\,L^{-1}t^{-1}]$ and time is measured in the same unit in both systems, it is necessary only to convert grams to

[1] This restriction is necessary, for as Prof. Bridgman pointed out in his *Dimensional Analysis* (Yale University Press, 1922 and 1931), the equations from physics, $v = at$ and $s = \frac{at^2}{2}$ may be added to form the equation $v + s = at + \frac{at^2}{2}$ which is not homogeneous. It is true, but of no use. The general requirement of homogeneity was apparently first stated by Fourier in his *Théorie Analytique de la Chaleur* (1822), Chap. II, Sec. 9.

slugs and then divide by the factor which converts cm. to ft. 1 gm. = 0.000068522 slugs and 1 cm. = 0.032808 ft.; therefore the conversion factor is $0.000068522 \div 0.032808 = 0.0020885$, and $0.01310 \times 0.0020885 = 0.00002735$, which is the required value.

EXAMPLE II

Check the dimensions in (49.3), $v_m = \frac{(p_1 - p_2)\, r_0^2}{4\, \mu\, L}$. The dimensions of p_1 and p_2 are $[\mathrm{m\, L^{-1}t^{-2}}]$ and $[\mathrm{r_0}] = [\mathrm{L}]$. Therefore the dimensions of the numerator are $[\mathrm{m\, L\, t^{-2}}]$. $[\mu] = [\mathrm{m\, L^{-1}t^{-1}}]$; therefore the dimensions of the denominator are $[\mathrm{m\, t^{-1}}]$ and the dimensions of the fraction are $[\mathrm{L\, t^{-1}}]$. These are also the dimensions of the left hand member, so that the equation is homogeneous.

PROBLEMS

68–1. If the density of water at 39.2° F. is exactly one in the c.g.s. system, what is it in the slug-ft.-sec. system? *Ans.* 1.940 slugs/cu. ft.

68–2. What acceleration in the c.g.s. system is equivalent to 32.174 in the slug-ft.-sec. system?

68–3. Show how the dimensions of the derived units in the m – L – t system listed above were derived.

68–4. Do the same for the F – L – t system.

68–5. If C has the dimensions $[\mathrm{L^{0.5}t^{-1}}]$, what value of C in the m.-kg.-sec. system will correspond to 100 in the ft.-lb.-sec. system?

68–6. Work Example II, using the F – L – t system.

68–7. Which of the following equations are certainly not homogeneous? In the case of the others, what must be the dimensions of the constant in order to make them homogeneous? (20.6), (25.2), (25.4), (25.7), (43.2), (43.3), (45.2), and those of Probs. 25–5, 25–11, and 25–15.

69. Buckingham's Π Theorem.—The fact that true equations in their simplest form must be dimensionally homogeneous can be used as a help in finding the proper form for empirical equations. If the magnitude of a physical quantity is a function of $n - 1$ other independent physical quantities, and if the n quantities involve altogether k fundamental units, Buckingham[2] showed that the equation connecting the n quantities

[2] "On Physically Similar Systems," *Physical Review*, Vol. 4 (1914), pp. 345–376, or better, "Model Experiments and the Forms of Empirical Equations," *Trans. A.S.M.E.*, Vol. 37 (1915), pp. 263–296. Although unknown in this country, practically the same principle had long before been published in France by A. Vaschy, *Annales télégraphiques* (Jan.-Feb., 1892), and *Comptes-Rendues de L'Académie des Sciences*, Vol. 114, p. 1416 and Vol. 115, p. 597.

(69.1) $$f(q_1, q_2, \ldots q_n) = 0$$

may be replaced by the equation

(69.2) $$\phi(\Pi_1, \Pi_2, \qquad \Pi_{n-k}) = 0,$$

where each of the Π's are independent dimensionless monomial functions of the q's (the observed physical quantities), any one Π depending on not more than $k + 1$ of the q's. The method of finding the Π's is illustrated in the examples. The first step is to list all the q's with their dimensions, and to note k, the number of fundamental units. Then k of the quantities are selected, no two of which have the same dimensions, and which among them include all the fundamental units. Then each Π is expressed as the product of each of these to an unknown power and one other q to a known power, and the unknown exponent solved for by the principle of dimensional homogeneity.

EXAMPLE I

Let it be supposed that the slope of the hydraulic gradient in a pipe depends only on the diameter and the mean velocity of flow. Find the type of formula to express this relationship.

Let $q_1 = D$ = diameter	[L]
$q_2 = V$ = mean velocity	$[\mathrm{L\,t^{-1}}]$
$q_3 = S$ = slope	$[\mathrm{L^0\,t^0}]$

We are seeking an equation of the form $f(D, V, S) = 0$. If we stop to think, we will realize that the quest is futile. We cannot have an equation of this sort, because V is the only quantity which contains the dimension of time, and to be homogeneous, the equation must contain at least one other term containing time, so that the time exponents can add up to zero. The thing to do in a case like this is to assume an additional q.

Let q_4 = acceleration of gravity $[\mathrm{L\,t^{-2}}]$. Now there are four q's and k is still two, so there will be two Π's. Selecting D and V as the two to receive unknown exponents, we have

(1) $$\Pi_1 : D^{x_1}V^{y_1}g$$
(2) $$\Pi_2 : D^{x_2}V^{y_2}S$$

The dimensional equation for (1) is $[\mathrm{L^0\,t^0}] = [\mathrm{L}^x]\,[\mathrm{L\,t^{-1}}]^y\,[\mathrm{L\,t^{-2}}]$. To make this homogeneous with respect to time requires that the sum of the exponents of t in the left-hand member equal the sum of those in the right-hand mem-

ber. Hence $0 = -y_1 - 2$ or $y_1 = -2$. Similarly, to make it homogeneous with respect to length, $0 = x_1 + y_1 + 1$ or $x_1 = 1$. Therefore $\Pi_1 = \frac{D\,g}{V^2}$. The dimensional equation for (2) is $[L^0\,t^0] = [L]^{x_2}[L\,t^{-1}]^{y_2}[L^0\,t^0]$. From t we have $0 = -y_2$ and from L, $0 = x_2 + y_2$ or $x_2 = 0$. Therefore $\Pi_2 = S$. This illustrates the general rule that whenever a q is dimensionless, it may be taken as one of the Π's without going through the routine procedure.

Equation (69.2) in this case is $\phi\left(\frac{D\,g}{V^2}, S\right) = 0$ or $S = \Phi\left(\frac{D\,g}{V^2}\right)$. This is as far as dimensional analysis will take us, but we know from Fanning's formula the proper form to give to the Φ function. It may be taken as $\frac{f\,V^2}{2\,D\,g}$. If (2) had been written $\Pi_2 = D^{x_2}V^{y_2}g^{-1}$, Π_2 would have come out $\frac{V^2}{D\,g}$. If it had been written $\Pi_2 = D^{x_2}V^{y_2}g^{-1/2}$, it would have been $\frac{V}{\sqrt{D\,g}}$. If an expression is dimensionless, any product or power of it is also dimensionless, therefore any Π may be replaced by any other power of it, including Π^{-1}, or by any product of it by a numerical constant, or by its product with any power of any other Π. As has already been mentioned in Art. 46, the expression $\frac{V}{\sqrt{D\,g}}$ occurs so often in hydraulics, that it has been given a symbol and a name. It is called the Froude number in honor of William Froude (English, 1810–1879) who was the pioneer experimenter on "surface friction," as well as "wave making resistance."

$$\mathbf{F} = \tag{69.3}$$

It typifies the effect of gravity in the same way that **R** typifies the effect of viscosity.[3]

[3] Actually gravity does not affect the pressure drop in a closed pipe. It comes into the problem only because we are measuring pressure drop by means of the corresponding head. As shown in Prob. 69–4, the dimensionless number here is really $\frac{\rho V^2}{\Delta p}$. This has no name or symbol but is the dimensionless number typifying inertia effects. See *Fluid Mechanics for Hydraulic Engineers* by Hunter Rouse, McGraw-Hill Book Co. (1938), Sec. 5.

EXAMPLE II

Work Example I including the effect of viscosity.

Viscosity includes a dimension of mass. Therefore, its inclusion requires the inclusion of another quantity including mass. The one that immediately suggests itself is density. This makes six quantities as follows:—

$q_1 = D =$ diameter	$[\mathrm{L}]$
$q_2 = V =$ mean velocity	$[\mathrm{L\ t^{-1}}]$
$q_3 = \rho =$ density	$[\mathrm{m\ L^{-3}}]$
$q_4 = g =$ acceleration of gravity	$[\mathrm{L\ t^{-2}}]$
$q_5 = \mu =$ viscosity	$[\mathrm{m\ L^{-1}\ t^{-1}}]$
$q_6 = S =$ slope	$[\mathrm{m^0\ L^0\ t^0}]$

Since k is now three, there will be three Π's, one of which is S. Let the other two be

$$(1) \qquad \Pi_1 = D^{x_1} V^{y_1} \rho^{z_1} g^{-1}$$
$$(2) \qquad \Pi_2 = D^{x_2} V^{y_2} \rho^{z_2} \mu^{-1}$$

The dimensional equation for (1) is $[\mathrm{m^0\ L^0\ t^0}] = [\mathrm{L}]^{x_1}[\mathrm{L\ t^{-1}}]^{y_1}[\mathrm{m\ L^{-3}}]^{z_1}[\mathrm{L\ t^{-2}}]^{-1}$.

Equating exponents of m, $\quad 0 = z_1$
Equating exponents of t, $\quad 0 = -y_1 + 2, \quad$ or $\quad y_1 = 2$
Equating exponents of L, $\quad 0 = x_1 + y_1 - 3z_1 - 1 \quad$ or $\quad x_1 = -1$

Therefore $\Pi_1 = \dfrac{V^2}{Dg} = \mathbf{F}^2$. It may be replaced by $\mathbf{F}$. The dimensional equation for (2) is $[\mathrm{m^0\ L^0\ t^0}] = [\mathrm{L}]^{x_2}[\mathrm{L\ t^{-1}}]^{y_2}[\mathrm{m\ L^{-3}}]^{z_2}[\mathrm{m\ L^{-1}t^{-1}}]^{-1}$.

Equating exponents of m, $\quad 0 = z_2 - 1, \quad$ or $\quad z_2 = 1$
Equating exponents of t, $\quad 0 = -y_2 + 1, \quad$ or $\quad y_2 = 1$
Equating exponents of L, $\quad 0 = x_2 + y_2 - 3z_2 + 1, \quad$ or $\quad x_2 = 1$

Therefore $\Pi_2 = \dfrac{D V \rho}{\mu} = \mathbf{R}$. Equation (69.2) for this problem becomes

$$\phi\,(S, \mathbf{F}, \mathbf{R}) = 0 \qquad \text{or} \qquad S = \Phi\,(\mathbf{F}, \mathbf{R})$$

Here again dimensional analysis must stop, but it has already told us quite a little. If the six quantities listed are the only ones which affect the problem, they need be known only to the extent that they affect S, $\mathbf{F}$, and $\mathbf{R}$. That is, we do not need to know μ and ρ, but only their ratio ν. And what is more important, it is not necessary to experiment through all the possible combinations of V, D, and ν. All that is necessary is to cover the desired range of values of $\mathbf{R}$, and this may be done by varying whichever

of the factors may be convenient. This is perhaps the most important single contribution of dimensional analysis to experimental hydraulics.

From our previous studies we know that the function Φ may be replaced by $\frac{V^2}{2\,g\,D}\,\psi\,(\mathbf{R})$ and that ψ is not a simple function of **R,** but a function of **R** and the relative roughness, as shown in Fig. 94. If ϵ had been included as one of the quantities, we would have had $\Pi_3 = D^{x_3}V^{y_3}\rho^{z_3}\epsilon$, with the dimensional equation $[\mathrm{m}^0\,\mathrm{L}^0\,\mathrm{t}^0] = [\mathrm{L}]^{x_3}[\mathrm{L}\,\mathrm{t}^{-1}]^{y_3}[\mathrm{m}\,\mathrm{L}^{-3}]^{z_3}[\mathrm{L}]$.

Equating exponents of m, $z_3 = 0$
Equating exponents of t, $y_3 = 0$, and
Equating exponents of L, $x_3 = -1$, therefore

$$\Pi_3 = \frac{\epsilon}{D}, \text{ which may be replaced by } \frac{\epsilon}{r_0} \text{ or } \frac{r_0}{\epsilon}$$

This illustrates the general rule that whenever two of the q's have the same dimensions, as in this case D and ϵ, their ratio may be taken as one of the Π's. If there are three, any two of the ratios may be taken, and so on.

Probably a word should be said as to the difference between this method and that used before Buckingham's time by Sir G. G. Stokes, James Thompson, Osborne Reynolds, and Lord Rayleigh. They assumed that (69.1) is a simple power function and solved for the unknown exponents. That is, to solve Example I, they would take $S = k\,D^xV^yg^z$. This gives the dimensional equation $[\mathrm{L}^0\,\mathrm{t}^0] = [\mathrm{L}]^x[\mathrm{L}\,\mathrm{t}^{-1}]^y[\mathrm{L}\,\mathrm{t}^{-2}]^z$, from which the exponential relationships of t give $0 = -y - 2z$, or $y = -2z$, and those of L give $0 = x + y + z$, or $x = z$. Therefore $S = \frac{k\,D^zg^z}{V^{2z}} = k\left(\frac{D\,g}{V^2}\right)^z$. Taking z as -1 and k as $\frac{f}{2}$ gives the same result as before. But if applied to Example II, it results in a series of straight lines instead of the curves of Fig. 94. In other words, the method is simpler than Buckingham's and gives the same result when the relationship is a simple exponential function, but it should not be depended upon, because we can never be sure that an unknown function is of this form.

PROBLEMS

69–1. Rework Example I, taking D and g as the quantities to have unknown exponents.

69–2. Rework Example II, taking D, g, and μ as the quantities to have unknown exponents.

69–3. What other combinations of three quantities could be given the unknown exponents in the preceding problem?

69–4. Let it be assumed that the pressure drop in a pipe depends only on its length, its diameter, the mean velocity of flow, and the density of the liquid. Find the dimensionless quantities to reduce the relationship to the form of (69.2).

69–5. Show that if the velocity of surface waves depends only on the depth of the liquid, and the acceleration of gravity, the formula must be of the form $V = k\sqrt{g\,D}$.

69–6. Find the form of the equation for the discharge from sharp triangular weirs, assuming that Q depends only on the angle of the weir, H, and g.

69–7. Solve the preceding, including the effect of viscosity and density.

69–8. Solve the preceding, including also the effect of surface tension.

69–9. Assuming that the discharge from an orifice depends only on the head, the area, the kinematic viscosity and surface tension of the liquid, and g; show that C_d should be a function of $\frac{A}{H^2}$, **R**, and **W**, where **R** is defined $\frac{\sqrt{A}\;\sqrt{2\,g\,H}}{\quad}$ and **W** as V·

70. Dimensionless Numbers.

—The Reynolds and Froude numbers are not the only ones that have received names. In some problems in the mechanics of liquids, as for example in water hammer, the bulk modulus of the liquid, or its apparent bulk modulus, K, must be considered. The dimensions of K are $[\mathrm{m\,L^{-1}t^{-2}}]$, therefore the Π resulting from combining this with D, V, and ρ, would have the form $\Pi = D^xV^y\rho^zK$. The dimensional equation would be $[\mathrm{m^0\,L^0\,t^0}] = [\mathrm{L}]^x[\mathrm{L\,t^{-1}}]^y[\mathrm{m\,L^{-3}}]^z[\mathrm{m\,L^{-1}\,t^{-2}}]$, which leads to $\Pi = \frac{K}{\rho\,V^2}\cdot$ The square root of the reciprocal of this is called the Cauchy number, in honor of Augustin Louis Cauchy (French, 1789–1857) who in 1829 showed that the rate of vibrations varied as the square root of the ratio of the modulus of elasticity to the density.

$$\mathbf{C} = V\sqrt{\frac{\rho}{K}} \tag{70.1}$$

Similarly, where surface tension is involved, we get another

dimensionless criterion, called the Weber number in honor of Moritz G. Weber (German, 1871–　　), who first stated it in 1919.

$$\mathbf{W} = V\sqrt{\frac{\rho L}{\sigma}} \tag{70.2}$$

where σ (sigma) stands for surface tension, and L is the important length dimension.

As pointed out by Dr. Rouse in *Fluid Mechanics for Hydraulic Engineers*, **F**, **C**, and **W** are, respectively, the ratios of the actual velocity of flow to the velocities of gravity, elastic, and capillary waves.

It is not necessary to repeat the methods of the preceding articles for every problem. If gravity affects the problem, **F** must be included; if viscosity, **R**; if elasticity, **C**; and if surface tension, **W**. Besides these, there are the ratios of the various length dimensions and $\frac{\rho V^2}{\Delta p}$. Unless pressure is to be included in the formula, the latter may be assimilated into **F**. So that, in general, hydraulic formulas will be of the form

$$= \phi\left(\mathbf{F}, \mathbf{R}, \mathbf{C}, \mathbf{W}, \frac{\rho V^2}{\Delta p}\ \frac{H}{D}, \frac{L}{D}, \frac{B}{D}\right) \tag{70.3}$$

usually with many of the terms lacking, or playing a very secondary part. It should also be added that many authors use **F, C,** and **W** to mean the square of the values defined above.

The rules for model similarity given in Chapter VIII may be rephrased in the light of our present discussion. If g is the same in model and prototype,

$$\mathbf{F}_r = \mathbf{F}_p \div \mathbf{F}_m = \frac{V_r}{D_r^{0.5}} \tag{70.4}$$

and preserving dynamic similarity, that is, keeping the velocity scale equal to the square root of the depth scale, is equivalent to keeping the Froude number constant. Since, if the liquid in the model has the same density as the liquid in the prototype,

this also makes $\dfrac{\rho V^2}{\Delta p}$ the same in model as in prototype, it is the most usual requirement of model similarity. Similarity which meets this requirement is sometimes called Froudian similarity.

As already pointed out in Chapter VIII, in model work where friction is appreciable, an attempt is sometimes made to keep Reynolds number the same in model and prototype, and as mentioned in Art. 63, this would lead to another set of scale ratios. These are given in Table XVII. Similar scale ratios for equal **C**'s and **W**'s for some of the quantities are listed on page 26 of Dr. Rouse's *Fluid Mechanics for Hydraulic Engineers*.

TABLE XVII

QUANTITY	SCALE RATIO FOR EQUAL:	
	F	R
Length	L_r	L_r
Area	L_r^2	L_r^2
Volume	L_r^3	L_r^3
Mass	$L_r^3 w_r g_r^{-1}$	$L_r^3 w_r g_r^{-1}$
Density	$w_r g_r^{-1}$	$w_r g_r^{-1}$
Time	$L_r^{0.5} g_r^{-0.5}$	$L_r^2 \nu_r^{-1}$
Velocity	$L_r^{0.5} g_r^{0.5}$	$L_r^{-1} \nu_r$
Acceleration	g_r	$L_r^{-3} \nu_r^2$
Angular velocity	$L_r^{-0.5} g_r^{0.5}$	$L_r^{-2} \nu_r$
Angular acceleration	$L_r^{-1} g_r$	$L_r^{-4} \nu_r^2$
Force and weight	$L_r^3 w_r$	$\nu_r^2 w_r g_r^{-1}$
Pressure	$L_r w_r$	$L_r^{-2} \nu_r^2 w_r g_r^{-1}$
Impulse and momentum	$L_r^{3.5} w_r g_r^{-0.5}$	$L_r^2 \nu_r w_r g_r^{-1}$
Discharge, volume per sec.	$L_r^{2.5} g_r^{0.5}$	$L_r \nu_r$
Discharge, weight per sec.	$L_r^{2.5} w_r g_r^{0.5}$	$L_r^{-2} \nu_r^3 w_r g_r^{-1}$
Energy and work	$L_r^4 w_r$	$L_r \nu_r^2 w_r g_r^{-1}$
Power	$L^{3.5} w_r g_r^{0.5}$	$L_r^{-1} \nu_r^3 w_r g_r^{-1}$
Torque	$L_r^4 w_r$	$L_r \nu_r^2 w_r g_r^{-1}$
Viscosity	$L_r^{1.5} w_r g_r^{-0.5}$	$\nu_r w_r g_r^{-1}$
Kinematic viscosity	$L_r^{1.5} g_r^{0.5}$	ν_r
Surface tension	$L_r^2 w_r$	$L_r^{-1} \nu_r^2 w_r g_r^{-1}$
Modulus of elasticity	$L_r w_r$	$L_r^{-2} \nu_r^2 w_r g_r^{-1}$

PROBLEMS

70–1. Show how the expressions for scale ratios in Table XVII for equal **R**'s for time, velocity, acceleration, and angular velocity and acceleration are derived.

70–2. Do the same for force, pressure, and discharge.

70–3. Do the same for momentum, energy, power, and torque.

70–4. Do the same for viscosity, kinematic viscosity, surface tension, and modulus of elasticity.

70–5. Find the necessary ratio of viscosity in prototype to viscosity in model to make both **F** and **R** the same in model and prototype, for discharge in volume per second and discharge in weight per second.

Ans. $L_r^{1.5}g_r^{0.5}$

70–6. Do the same for the last five quantities in the table.

APPENDIX C

RATIONAL BASIS FOR NIKURADSE'S FORMULAS

71. The Nature of Turbulence.—In turbulent flow in a pipe, each particle of liquid will, in general, have besides its forward velocity, a velocity component toward or away from the center of the pipe, and a third component perpendicular to the other two. At any one point in the pipe these velocity components vary with the time in a rapid and irregular manner, so that inward velocity may change to outward velocity and back again several times a second. Also, the forward component varies in amount, perhaps sometimes becoming negative. It is probably best thought of as a uniform forward velocity v, with a constantly varying velocity superimposed on it. Over any appreciable time, say a few seconds, this last mentioned velocity, as well as the radial and circumferential ones mentioned above, will, if the flow is steady, have an average value of zero. Their effect, or at least the effect of the radial component, cannot be neglected, however.

As has been pointed out several times before, velocity ordinarily varies across the cross-section, with the highest velocity at the center. Therefore, if any liquid moves from the central part toward the walls, it will be moving into a slower moving region, and by its inertia will tend to speed up that region, and will at the same time be slowed up itself. This is equivalent to a shearing stress on each cylindrical surface, forward on the liquid outside, and backward on the liquid inside. On the other hand, if liquid moves from the outer regions toward the center it will tend to reduce the average velocity, and therefore acts as a drag on the inner portions. This also is equivalent to a shearing stress on each concentric cylindrical surface, backward on the inside portion, and forward on the outside portion. This stress, like that in laminar flow, can be shown to vary as the rate of change of velocity with distance from the wall, so that the stress due to the "mixing" can be written as $\eta \frac{dv}{dy}$, where η (eta), is called the coefficient of

mechanical viscosity. Since viscous shear as given by (48.3) is still operative in turbulent flow, the equation that gives the total effective shearing stress at any distance y from the wall, greater than the thickness of the boundary layer, is

$$\tau = \mu \frac{dv}{dy} + \eta \frac{dv}{dy} \tag{71.1}$$

Unlike ordinary viscosity, η is not simply a property of the liquid, but depends upon the flow conditions and the distance from the center of the pipe. Since η is generally at least 100 times as much as μ, the latter is generally neglected in turbulent flow.

This conception was first published by Boussinesq in 1877, and was much elaborated by Ludwig Prandtl and von Karman in the twenties and early thirties.[1] The latter seems not entirely satisfied with this conception based on "momentum transfer" and is now trying to develop a theory based on mathematical probability in somewhat the same way as is the kinetic theory of gases.[2]

We will not attempt to expound any of these theories here, but will base our derivation on dimensional reasoning and the results of experiments.

72. Smooth Pipes.—Assume that the velocity at a distance y from the wall of a smooth pipe depends only upon the viscosity and density of the fluid, the shearing stress at the wall, and y. Then the five quantities have the following dimensions:

$$\begin{aligned}
[\tau_0] &= [\mathrm{m\,L^{-1}\,t^{-2}}] \\
[\rho] &= [\mathrm{m\,L^{-3}}] \\
[\mu] &= [\mathrm{m\,L^{-1}\,t^{-1}}] \\
[\mathrm{v}] &= [\mathrm{L\,t^{-1}}] \\
[\mathrm{y}] &= [\mathrm{L}]
\end{aligned}$$

[1] A very complete explanation of these ideas is given in *The Mechanics of Turbulent Flow* by Boris A. Bakhmeteff, Princeton University Press, 1936. A somewhat more condensed account is given in "Modern Conceptions of Fluid Turbulence" by Hunter Rouse, *Trans. A.S.C.E.*, Vol. 102 (1937), pp. 463–543.

[2] "On the Statistical Theory of Isotropic Turbulence" by Theodore de Karman and Leslie Howarth, *Proc. Royal Soc.*, Vol. 164 (Jan., 1938), pp. 192–215. For the work of others on the same problem see "Statistical Theory of Isotropic Turbulence" by G. I. Taylor, *Jour. of the Aeronautical Sci.*, Vol. 4 (1937), pp. 311–315; and "The Theory of Isotropic Turbulence" by Hugh L. Dryden, *Jour. of the Aeronautical Sci.*, Vol. 4 (1937), pp. 273–280.

There are three fundamental units, so there will be two dimensionless numbers involved. Let them be

$$\Pi_1 = \tau_0^{x_1}\rho^{y_1}\mu^{z_1}v \qquad \text{and}$$
$$\Pi_2 = \tau_0^{x_2}\rho^{y_2}\mu^{z_2}y$$

From the first, $[\mathrm{m}^0\mathrm{L}^0\mathrm{t}^0] = [\mathrm{m\,L^{-1}t^{-2}}]^{x_1}[\mathrm{m\,L^{-3}}]^{y_1}[\mathrm{m\,L^{-1}\,t^{-1}}]^{z_1}[\mathrm{L\,t^{-1}}]$

Equating the exponents of m, $\quad 0 = x_1 + y_1 + z_1$
Equating the exponents of L, $\quad 0 = -x_1 - 3\,y_1 - z_1 + 1$
Equating the exponents of t, $\quad 0 = -2\,x_1 - z_1 - 1.$
Solving simultaneously, $\quad x_1 = -0.5,\ y_1 = 0.5,\ z_1 = 0,\quad$ and

$$\Pi_1 = \frac{\rho^{0.5}v}{\tau_0^{0.5}}$$

$$\sqrt{\frac{\tau_0}{\rho}}$$

From the second equation,

$$[\mathrm{m}^0\,\mathrm{L}^0\,\mathrm{t}^0] = [\mathrm{m\,L^{-1}\,t^{-2}}]^{x_2}[\mathrm{m\,L^{-3}}]^{y_2}[\mathrm{m\,L^{-1}\,t^{-1}}]^{z_2}[\mathrm{L}]$$

Equating the exponents of m, $\quad 0 = x_2 + y_2 + z_2$
Equating the exponents of L, $\quad 0 = -x_2 - 3\,y_2 - z_2 + 1.$
Equating the exponents of t, $\quad 0 = -2\,x_2 - z_2$
Solving simultaneously, $\quad x_2 = 0.5,\ y_2 = 0.5,\ z_2 = -1,\quad$ and

$$\Pi_2 = \frac{\tau_0^{0.5}\rho^{0.5}y}{\mu} = \frac{\rho\,y}{\mu}\sqrt{\frac{\tau_0}{\rho}} = \frac{y}{\nu}\sqrt{\frac{\tau_0}{\rho}}$$

The expression $\sqrt{\frac{\tau_0}{\rho}}$ has the dimensions of a velocity and has been given the name of *friction velocity*. It is represented by the symbol u_*. Taking the liquid in length L of a pipe of diameter D as the free body, we have $\tau_0\pi\,D\,L = (p_1 - p_2)\,\frac{\pi\,D^2}{4} = w\,h_f\,\frac{\pi\,D^2}{4}$, from which $h_f = \frac{4\,\tau_0 L}{w\,D} = \frac{4\,\tau_0 L}{g\,\rho\,D}\cdot$ Equating this to $\frac{f\,L\,V^2}{D\,2\,g}$, gives $\frac{f\,V^2}{2} = \frac{4\,\tau_0}{\rho}$, from which

$$u_* = \sqrt{\frac{\tau_0}{\rho}} = V\sqrt{\frac{f}{8}} \qquad (72.1)$$

That is, the friction velocity in a pipe is $\sqrt{\frac{f}{8}}$ times the mean velocity V. For example, when $f = 0.020$, $\frac{f}{8} = 0.0025$, $u_* = 0.05\ V$ and $\frac{V}{u_*} = 20$.

Returning to our dimensional analysis from this digression, we have the relationship

$$\frac{v}{u_*} = \phi\left(\frac{y\,u_*}{\nu}\right) \tag{72.2}$$

It will be noted that both sides of this equation are dimensionless, and that the quantity inside the parenthesis is of the general form

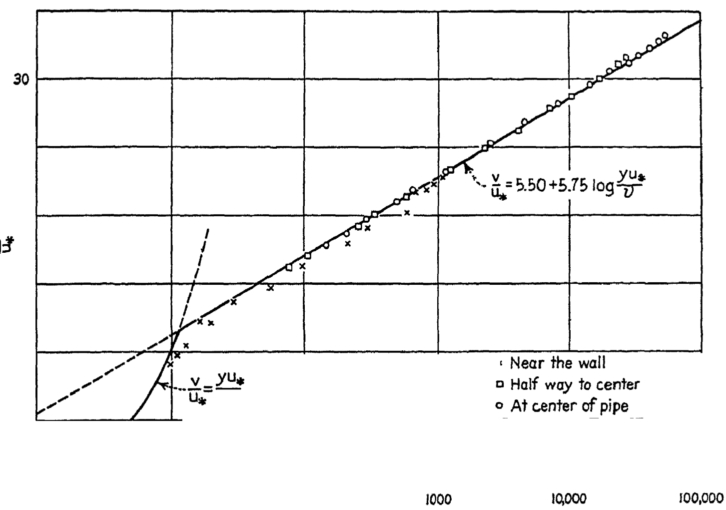


FIG. 122.—Variation of "friction velocity" in smooth pipes with $\frac{y\,u_*}{\nu}$.

of a Reynolds number, the length being y and the velocity u_*. To find the form of the ϕ function, we take Nikuradse's experimental results as plotted in Fig. 122, using a true scale for $\frac{v}{u_*}$ and a logarithmic scale for $\frac{y\,u_*}{\nu}$. It will be seen that most of the points

lie fairly close to the line

$$\frac{v}{u_*} = 5.50 + 5.75 \log \frac{y\, u_*}{\nu} \tag{72.3}$$

(the logarithm being to the base 10) and that the maximum velocity, which occurs when $y = r_0$, will be

$$\frac{v_m}{u_*} = 5.50 + 5.75 \log \frac{r_0 u_*}{\nu} \tag{72.4}$$

Equation (72.3) does not hold clear to the wall, as it gives a zero value of v before y is zero, and $v = -\infty$ for $y = 0$. This need not trouble us, however, as the flow is not turbulent next to the wall. In the laminar layer, $\tau = \mu \frac{dv}{dy}$, and through this thin layer the shear may be assumed constant and the variation of v with y taken as linear, so that $\tau_0 = \frac{\mu\, v}{y} = \frac{\nu\, \rho\, v}{y}$. Then $\frac{\nu\, v}{y} = \frac{\tau_0}{\rho} = u_*^2$, which can be rearranged into $\frac{v}{u_*} = \frac{y\, u_*}{\nu}$. If a true scale had been used in Fig. 122, this would have plotted as a straight line, but with the semilogarithmic scale it plots as a curve as shown. It intersects the line for turbulent flow at $\frac{v}{u_*} = \frac{y\, u_*}{\nu} = 11.6$. This intersection is quite naturally taken as the division between laminar and turbulent flow, and the corresponding value of y is taken as the thickness of the boundary layer and represented by δ. This gives $y = \delta = 11.6\, \nu \quad \frac{11.6\, \nu}{V\sqrt{f/8}} - \frac{11.6\, D}{\mathbf{R}\sqrt{f/8}}$, which is (53.1).

Subtracting (72.3) from (72.4) we get

$$v = 5.75\, u_* \log_{10} \frac{r_0}{y} \quad 2.50\, u_* \ln \frac{r_0}{y} \tag{72.5}$$

where ln indicates the natural or Naperian logarithm. This formula was derived by Prandtl in 1933, by a more deductive method than we have used.

As will be seen from Fig. 77, the discharge,

$$Q = \pi r_0^2 V = \pi r_0^2 v_m - \int_0^{r_0} (v_m - v)\, 2\pi\, (r_0 - y)\, dy.$$[3]

Substituting from (72.5), this last integral becomes

$$5.00\, \pi\, u_* \int_0^{r_0} (r_0 - y) \ln \frac{r_0}{y}\, dy$$

$$= 5\, \pi\, u_* \int_0^{r_0} (r_0 \ln r_0 - r_0 \ln y - y \ln r_0 + y \ln y)\, dy$$

$$= 5\, \pi\, u_* \left[y\, r_0 \ln r_0 - r_0 (y \ln y - y) - \frac{y^2}{2} \ln r_0 + y^2 \left(\frac{\ln y}{2} - \frac{1}{4} \right) \right]_0^{r_0}$$

$$= 5\, \pi\, u_* \left[r_0^2 \ln r_0 - r_0^2 \ln r_0 + r_0^2 - \frac{r_0^2}{2} \ln r_0 + \frac{r_0^2}{2} \ln r_0 - \frac{r_0^2}{4} \right]$$

$$= 3.75\, \pi\, u_* r_0^2$$

Then $Q = \pi r_0^2 v_m - 3.75\, \pi\, u_* r_0^2$ and

$$V = v_m - 3.75\, u_* \tag{72.6}$$

Substituting in this the value of v_m from (72.4), we have

$$= u_* \left(1.75 + 5.75 \log \frac{r_0 u_*}{\nu} \right)$$

Substituting in this the value of u_* from (72.1), we have

$$1 = \sqrt{\frac{f}{8}} \left(1.75 + 5.75 \log \frac{r_0 V}{\nu} \sqrt{\frac{f}{8}} \right),$$

and
$$\frac{1}{\sqrt{f}} = 0.619 + 2.03 \log \frac{D\, V \sqrt{f}}{\nu \sqrt{32}}$$

$= 0.619 + 2.03 \log \mathbf{R} \sqrt{f} - 2.03 \log \sqrt{32} = 2.03 \log \mathbf{R} \sqrt{f} - 0.91.$
However, Nikuradse found that a change of the constants to

[3] There is, of course, an approximation involved in integrating clear to the wall. As pointed out above, (72.3) does not apply for values of y less than δ. Integrating from δ to r_0 and then adding the discharge in the boundary layer gives a somewhat different result—how much different depends on the Reynolds number. At high Reynolds numbers, the error is less than one per cent. At low Reynolds numbers, it may be several per cent.

(51.2)
$$\frac{1}{\sqrt{f}} = 2 \log \mathrm{R} \sqrt{f} - 0.80$$

gave better agreement with his discharge measurements. This small discrepancy is probably one of the reasons that leads von Karman and others to seek a still better formula.

PROBLEMS

72–1. Show that u_* has the dimensions of velocity.

72–2. Solve the simultaneous equations and check the value of Π_1.

72–3. Do the same for Π_2.

72–4. Show in detail the derivation of $\frac{v}{u_*} = \frac{y\,u_*}{\nu}$.

72–5. Solving as indicated in Note 3, show that when $\delta = 0.001\,r_0$, the 3.75 obtained above becomes 3.71.

73. Rough Pipes.—If it is assumed that the velocity at y distance from a rough pipe wall depends only on the equivalent roughness ϵ, the shear at the wall, the density of the liquid, and y, dimensional analysis leads to the equation

(73.1)
$$v = u_*\psi\left(\frac{y}{\epsilon}\right.$$

A plotting of the results of Nikuradse's experiments shows that the ψ function can be well represented by the expression $8.48 + 5.75 \log \frac{y}{\epsilon}$.[4] Then

(73.2)
$$v = u_*\left(8.48 + 5.75 \log \frac{y}{\epsilon}\right),$$

(73.3)
$$v_m = u_*\left(8.48 + 5.75 \log \frac{r_0}{\epsilon}\right); \text{ and}$$

(73.4)
$$v_m - v = 5.75\,u_* \log \frac{r_0}{y} = 2.50\,u_* \ln \frac{r_0}{y}$$

[4] See, for example, Fig. 18 in "Mechanics of Fluid Turbulence," *Trans. A.S.C.E.*, Vol. 102 (1937), p. 498. The figure 8.48 was there erroneously printed as 5.85, and this error was carried forward into *Fluid Mechanics for Hydraulic Engineers*, Fig. 118, p. 243.

which is identical with (72.5). Just as in Art. 72, this equation leads to (72.6). Adding (72.6) and (73.3) gives

$$= u_*\left(8.48 + 5.75 \log \frac{r_0}{\epsilon} - 3.75\right), \text{ or}$$

$$= u_*\left(4.73 + 5.75 \log \frac{r_0}{\epsilon}\right) \tag{73.5}$$

Substituting (72.1) in this expression and solving gives

$$\frac{1}{\sqrt{f}} = 1.67 + 2.03 \log \frac{r_0}{\epsilon},$$

which with a slight adjustment of the coefficients better to fit the discharge measurements, gives

$$\frac{1}{\sqrt{f}} = 1.74 + 2 \log \frac{r_0}{\epsilon} \tag{73.6}$$

which is the same as (30.1).

PROBLEMS

73–1. Work out in detail the dimensional analysis to derive (73.1).

73–2. Work out in detail the steps in the derivation of (30.1) from (72.6) and (73.2).

APPENDIX D

STREAM GAGING

74. Stream Gaging.—The usual method of measuring the flow of water in a stream, is with a *current meter*. See Fig. 123. This is an instrument with cups or vanes which revolve at a rate proportional to the velocity of the water. (Current meters are calibrated by towing them through still water in a rating flume in the U. S. Bureau of Standards.) At each revolution (or sometimes at each second or fifth revolution) an electric contact is broken, which makes a click in the observer's head-phone. With a stop-watch, he finds the time for a certain number of revolutions. The meter is free to turn about a vertical axis, and has a tail which automatically puts it in line with the current. For low water in small streams the gager wades in hip-boots and supports the meter on a rod which is marked in tenths of feet. See Fig. 123. These rods are also used for working through ice on shallow rivers, and sometimes when working from low bridges. For deeper water, the gager works from a bridge, a cable-way especially built for the purpose (see Figs. 87 and 124), or (sometimes) from a boat. The meter is held by a wire "line" and a stream-lined weight is hung below the meter to keep it from being carried too far down-

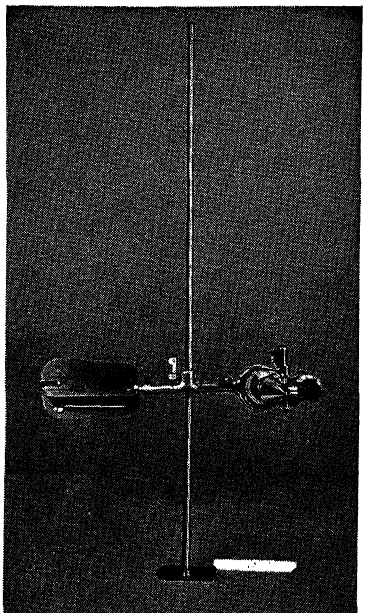


FIG. 123.—Improved Price current meter with wading rod suspension. (Courtesy of U. S. Geological Survey.)

stream by the current. For 15 or 30 pound weights a "hand line" may be used, but for 50, 75, 100, or 150 pound weights, such as are required in deeper or swifter rivers, and especially in flood time, a reel is used. This is designed so that the length of line let out is shown to the tenth of a foot on a revolution counter attached to the reel. By noting the reading when the weight first touches the water, and again when it touches bottom, the depth is found. However, if it is carried an appreciable amount downstream, a correction is necessary. (Also, if the current is not at right angles to the cross-section a correction to the velocity must be made.) For use in high water on our largest rivers weights of 200, 300, and even 500 pounds are used. To operate them, a power driven reel mounted on an especially built automobile has been developed in the Columbus, Ohio, office of the United States Geological Survey.

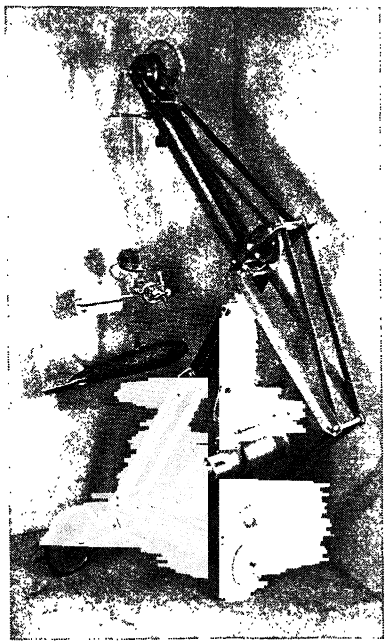


Fig. 124.—Current meter with 75 lb. sounding weight, crane and reel, used for measurements from bridges. (Courtesy of U. S. Geological Survey.)

When the velocity is measured at a number of different depths, and these values plotted against the depth, we get a vertical velocity curve. Such curves are found to be practically all parabolas, with the maximum velocity somewhere in the upper third of the depth. The velocity at the surface is generally more than the average velocity, but this is affected by the wind. There is always one depth at which the velocity equals the average velocity, and it is usually at about 0.6 the depth. However, it has been found that a more accurate method of gaging the flow is to take the mean velocity as the aver-

age of the velocity at 0.2 the depth and at 0.8 the depth, and these are the positions at which the velocity is usually measured. Since the center of the meter cannot be brought closer than a certain distance from the bottom, 0.8 readings cannot be taken in too shallow water; only 0.6 or surface readings are then taken.

The example gives an illustration of an actual gaging in a medium sized stream at low water (by wading). A "tag line" with markers every five feet was stretched across the stream, and depth and velocity taken each 5 feet. In the shallowest water the velocity was measured at the surface (marked S), and the mean velocity taken as 0.9 the surface velocity. In computing the discharge, the cross-sectional area is thought of as divided into partial areas by the verticals in which measurements were taken. Each partial area is taken to be the average of the depths on each side, multiplied by the width between measurements. The average of the "mean vertical" velocities at the two sides, multiplied by the partial area, gives the partial discharge, and the sum of these is the total discharge.

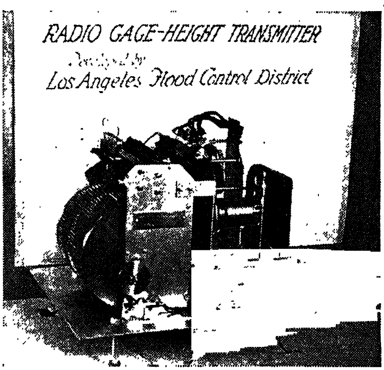


Fig. 125.—Instrument sends signals every 15 minutes. By counting buzzes, office can read gage height at distant station. For further description see *Eng. News-Record*, Jan. 19, 1939, p. 55. (Courtesy of U. S. Geological Survey.)

This example hardly comes up to present practice of the United States Geological Survey in regard to the number of sections. The rule now is that not more than 10 per cent of the total flow shall be in any one section, and preferably not more than 5 per cent. To insure this, it is best to take a minimum of 30 sections. To meet this standard, the gaging in the example should have had a spacing of 3 feet, at least in the central portion. Laboratory tests have shown that the current meter can be depended on to give results within 2 per cent of the truth if the velocity is not less than 1.5 feet per second, and the flow is

EXAMPLE

A Gaging of the Scioto River at Columbus, July 23, 1936

Dist. from Init. Point	Depth	Depth of Obs.	Revolu-tions	Time in Sec.	Velocity: At Point	Velocity: Mean Vert.	Velocity: Mean in Sect.	Width	Mean Depth	Area	Dis-charge
5	0.30	S	0		0	0					
							.22	5	.40	2.00	0.44
10	0.50	S	10	49	.48	.43					
							.44	5	.575	2.88	1.27
15	0.65	.6	10	54	.44	.44					
							.48	5	.875	4.38	2.10
20	1.10	.6	10	45	.52	.52					
							.55	5	1.15	5.75	3.16
25	1.20	.6	10	40	.58	.58					
							.57	5	1.40	7.00	3.99
30	1.60	.6	10	41	.56	.56					
							.57	5	1.75	8.75	4.99
35	1.90	.2	15	45	.77	.58					
		.8	10	60	.40		.61	5	2.00	10.00	6.10
40	2.10	.2	20	60	.77	.64					
		.8	10	47	.50		.63	5	2.02	10.10	6.36
45	2.10	.2	15	47	.74	.62					
		.8	15	73	.49		.63	5	2.02	10.10	6.36
50	1.95	.2	15	48	.72	.64					
		.8	10	41	.56		.64	5	2.00	10.00	6.40
55	2.05	.2	15	47	.74	.63					
		.8	10	45	.52		.63	5	1.98	9.90	6.24
60	1.90	.2	15	49	.71	.63					
		.8	10	42	.55		.66	5	1.80	9.00	5.94
65	1.70	.6	15	50	.69	.69					
							.68	5	1.58	7.90	5.37
70	1.45	.6	15	52	.67	.67					
							.66	5	1.32	6.60	4.36
75	1.18	.6	15	55	.64	.64					
							.58	5	1.04	5.20	3.02
80	0.90	.6	10	44	.52	.52					
							.52	5	.80	4.00	2.08
85	0.70	.6	10	45	.51	.51					
							.54	5	.68	3.40	1.84
90	0.65	S	15	53	.65	.58					
							.48	5	.62	3.10	1.49
95	0.60	S	10	57	.42	.38					
							.19	5	.50	2.50	0.48
100	0.40	S	0		0	0					
								Total Discharge =			72.25

straightforward with little turbulence.[1] It seems, considering all sources of error, that the probable error of current meter gagings taken as described above, will run about 3 per cent when conditions are favorable, and considerably higher when they are unfavorable.

PROBLEMS

74-1. Make a drawing of the cross-section of the example, using a scale of 1″ = 10′ horizontally, and 2″ = 1′ vertically, and write the mean velocity, and the discharge, in each sub-area.

74-2. Check the computations of the example.

74-3. Compute the discharge from the following data. (From 30 ft. on readings were taken every 3 ft., but alternate values are omitted here.)

Distance from Initial Point	Depth	Mean Velocity in Vertical
4	0	0
10	0.20	0.17
15	0.35	0.34
20	0.58	0.93
25	0.95	1.01
30	1.02	1.01
36	1.50	1.16
42	1.70	1.10
48	2.10	1.03
54	2.38	1.12
60	2.32	1.19
66	2.40	1.20
72	2.36	1.22
78	2.22	1.25
84	1.98	1.22
90	1.55	1.26
96	1.19	1.18
102	0.89	1.12
108	0.78	0.97
114	0.75	0.38
116	0	0

Ans. Using all of the data, the U. S. Geological Survey computed the discharge as 179.51 c.f.s.

[1] See "Effect of Turbulence on the Registration of Current Meters," by David L. Yarnell and Floyd A. Nagler, *Trans. A.S.C.E.*, Vol. 95 (1931), pp. 766–860.

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